Transactions - The Royal Society Of Edinburgh, Volume 39, Part 2

Scholar SELECT

ROYAL SOCIETY OF EDINBURGH

Transactions - The Royal Society Of Edinburgh, Volume 39, Part 2

TRANSACTIONS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. XXXIX. PART II.—FOR THE SESSION 1897–98.

CONTENTS.

EDINBURGH:

PUBLISHED BY ROBERT GRANT & SON, 107 PRINCES STREET,

AND WILLIAMS & NORGATE, 14 HENRIETTA STREET, COVENT GARDEN, LONDON.

MDCCCXCIX.

Price Nineteen Shillings.

IX.—*On the Definite Integral* $\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$, *with Extended Tables of Values.*

By Jas. Burgess, C.I.E., LL.D.

(Read July 15, 1895.)

1. The integral $\int e^{-t^2}dt$ occurs so frequently in various branches of research that, as far back as 1783, Laplace suggested that it would be useful to tabulate its values for successive ranges of integration.* It is employed in investigations on the theories of refraction, conduction of heat, of errors of observation, of probabilities, etc. These are familiar to physicists and need not be dwelt upon.†

The Integral.—Previous Tables.

2. The important formula or result—

$$\int_0^\infty e^{-t^2}dt = \tfrac{1}{2}\sqrt{\pi}\,. \tag{1}$$

appears to have been discovered about 1730 by Euler,‡ who expressed it in the form—

$$\int_0^1 \left(\log_e\frac{1}{x}\right)^{-\frac{1}{2}} dx = \sqrt{\pi}\,;\S$$

for, putting $x = e^{-t^2}$, we have $\left(\log_e\frac{1}{x}\right)^{-\frac{1}{2}}dx = -2e^{-t^2}dt$.

3. Since

$$\int_0^\infty e^{-t^2}dt = \int_0^t e^{-t^2}dt + \int_t^\infty e^{-t^2}dt\,, \tag{2}$$

* *Histoire de l'Acad. Roy. des Sciences*, 1783, p. 434; conf. Todhunter, *Hist. of the Theory of Probabilities*, p. 486.

† Conf. Glaisher, in *Phil. Mag.*, vol. xlii, (1871), pp. 429–31.

‡ Gauss ascribed this integration to Laplace; Oriani (in Zach's *Monatliche Corresp.* for March 1810, Bd. xxi, S. 280 f.) pointed out Euler's prior claim, but Gauss did not correct his statement, *Theoria Motus Corp. Cœl.*, art. 177, p. 212, and *Werke*, Bd. vii, Ss. 233, 280, 289; Davis's transl. of *Theor. Mot.*, pp. 258, 259. Legendre (*Exercices de Calcul Intégral* (1811), tom. i, p. 301) asserts Euler's discovery, and refers to his paper, "Evolutio formulæ integralis $\int x^{f-1}dx(lx)^{\frac{m}{n}}$," in *Novi Commentarii Acad. Scient. Imp. Petropol.*, tom. xvi, (for 1771) p. 111. Conf. *ib.*, p. 101; and *Comment. Acad. Scient. Petrop.*, tom. v, (for 1730–1731) p. 44; also Euler's letter to Goldbach of 8th Jan. 1730, in Fuss, *Correspond. Math. et Phys.*, tom. i, p. 13.

§ This is the form used by Legendre in his "Traité des Intégrales Eulériennes" in *Fonctions Elliptiques*, etc. tom. ii, pp. 365, 517–524.

the integral may be taken as separated into two parts—

(1) $\int_0^t \epsilon^{-t^2} dt$, which Mr J. W. L. GLAISHER calls the *Error-function complement*, and indicates by 'Erfc.' And—

(2) $\int_t^\infty \epsilon^{-t^2} dt$, which Mr GLAISHER proposes to call the *Error-function*, denoting it by 'Erf.'* Mr R. PENDLEBURY accepts Mr GLAISHER's name for the second, and writes the first as 'erf,'—which might lead to mistakes.† For convenience of reference, we may indicate the second by G.

And we shall put for the multiple of the first function here dealt with—

$$H = \frac{2}{\sqrt{\pi}} \int_0^t \epsilon^{-t^2} dt. \tag{3}$$

Whence, from (2)— $$H = 1 - \frac{2}{\sqrt{\pi}} G ; \text{ and } G = \frac{\sqrt{\pi}}{2} (1 - H). \tag{4}$$

And since $\frac{1}{2}\sqrt{\pi} = 0.886\,226\,925\,452\,758\,013\,649\,083\,741\,670$,

and its reciprocal— $\frac{2}{\sqrt{\pi}} = 1.128\,379\,167\,095\,512\,573\,896\,158\,903\,120$,

also $\log \frac{2}{\sqrt{\pi}} = 0.052\,455\,059\,316\,914\,268\,038\,104\,750\,579$,

it is comparatively easy to derive the value of G from that of H, or the converse.

4. In 1789, M. KRAMP, in his *Analyse des Réfractions*, was the first to tabulate G from $t = 0.00$ to $t = 3.00$, for every hundredth of a unit, together with the logarithmic values and differences. To these he added a third table of the logarithmic values of $\epsilon^{t^2} G = \epsilon^{t^2} \int_t^\infty \epsilon^{-t^2} dt$, which is useful in connection with the theory of refraction. KRAMP apparently computed the earlier part of his table by the usual formula (8) given below; but it converges so slowly for values of $t > 1$, that KRAMP employed a difference formula —to be referred to later—in order to fill up and complete his table. For the lower values of t his results are carried to eight places, and are generally quite accurate; from $t = 2$ to $t = 3$ the values are carried to eleven places, and for the last he gives G = ·00001957729 in the table and ·00001957669 in the text,‡—the true value being ·00001957719 3236779.

BESSEL, in discussing the theory of refraction in his *Fundamenta Astronomiæ* (1818), pp. 36, 37, next gave two tables,§ the first of log. $\epsilon^{t^2} \int_t^\infty \epsilon^{-t^2} dt$ from $t = 0$ to $t = 1.00$,

* *Philos. Mag.*, vol. xlii., 4th ser. (1871), pp. 296, 297, 421.

† *Ibid.*, p. 437. If either is to be called "Error-function," it would seem to apply rather to H than to G.

‡ Twice, pp. 134, 135.

§ In March 1816 appeared GAUSS' *Bestimmung der Genauigkeit der Beobachtungen*, in which he employs several of the constants dependent on values of H.—*Werke*, Bd. iv, Ss. 110, 111, 116.

agreeing in the main with KRAMP's third table, but differing occasionally in the last, or 7th, figure. This may have been due to some recomputation in places where the third differences were irregular. His second table is a continuation of the first, employing as arguments $\log_{10}x$, from 0 to 1 at intervals of ·01, with first and second differences. This is equivalent to a short table of $\log_{10}(\epsilon^{t^2}G)$ from $t=1$ to $t=10$, arranged at intervals in a geometrical proportion of which the ratio is—

$$t \times \log^{-1}\cdot 01 = t \times 1\cdot 023\ 292\ 992\ 281.$$

It is not explained how this table was computed.

The next table of the kind appeared in LEGENDRE's "Intégrales Eulériennes" (1826),* giving 130 values of 2G, computed to ten decimal places, and arranged in two parts. The first contains the values from $t=0\cdot 00$ to $t=0\cdot 50$, computed by the usual series, and by halving the values of the integrals we can readily verify or correct the early part of KRAMP's first Table. The second part is adapted to EULER's form of the integral, viz.—

$$\int\left(\log_e\frac{1}{x}\right)^{-\frac{1}{2}}dx: \qquad t=\left(\log_e\frac{1}{x}\right)^{\frac{1}{2}},$$

and is arranged with x as argument, from $x=0\cdot 80$ (that is, $t=0\cdot 472\ 380\ 727\ 077$) to $x=0\cdot 00$ or $t=\infty$. But, though when $x=0$, t is infinite,—in the previous entry, $x=0\cdot 01$ makes $t=\sqrt{\log_e 100}=2\cdot 145\ 966\ 026\ 289$,—so that this table does not really cover the extent of KRAMP's. It was computed by quadratures, and the process is laborious and effected by means of logarithmic tables extended to twelve decimal places.†

In his "Theory of Probabilities" ‡ (1837), DE MORGAN reproduced KRAMP's table of this integral (G) without revision. Mr GLAISHER, in the *Philosophical Magazine* for December 1871,§ has further extended it from $t=3\cdot 0$ to $t=4\cdot 5$ at intervals of 0·01, to eleven places for the first fifty values, thirteen for the next, and fourteen for the last fifty. It would be easy enough to compute it in the way indicated below for any higher values of the argument (§ 25).

5. But it is with the other integral that this paper is concerned, viz.—

$$H=\frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2}dt = 1-\frac{2}{\sqrt{\pi}}\int_t^\infty \epsilon^{-t^2}dt.$$

A table of this was first published by ENCKE, in a paper on the Method of Least Squares, in the *Berliner Astronomisches Jahrbuch* for 1834,|| giving the values of the integral, for the arguments $t=0$ to $t=2\cdot 00$ at intervals of 0·01, computed to seven decimal places, with first and second differences. This table, the author says, was derived

* In his *Traité des Fonctions Elliptiques et des Integrales Eulériennes*, tom. ii, pp. 520, 521.

† *Op. cit.*, tom. ii, pp. 517–524. The method explained below (§ 12) is different.

‡ In *Encyclopædia Metropolitana*, vol. ii, pp. 359–458. He also gave a short abstract of it in his *Differential and Integral Calculus* (1842), p. 657.

§ Vol. xlii, 4th ser., p. 436.

|| The paper is continued through the vols. for 1834 (Ss. 249–312), 1835 (253–320), and 1836 (253–308). The Table is in the *Jahrbuch* for 1834, Ss. 305–308.

immediately from the table for the integral $\int e^{-t^2}dt$ in BESSEL's *Fundamenta Astronomiæ*.* There seems to be a mistake here, for the table could be derived directly only from KRAMP's Table I.

DE MORGAN reproduced this table also in his "Theory of Probabilities" (*Encyclop. Metrop.*, 1837), and again in his *Essay on Probabilities* (1838), but there he extended it to $t=3{\cdot}00$ from KRAMP's data. Again, GALLOWAY, in his "Treatise on Probability" (1839), prepared for the 7th edition of the *Encyclopædia Britannica*, printed ENCKE's Table, also continued to the same point.

6. Further, and in dependence upon this integral, ENCKE gave a table † of the values of

$$\frac{2}{\sqrt{\pi}}\int_0^{\rho t} e^{-t^2}dt = \mathrm{K}, \tag{5}$$

ρ being the numerical value of t when $\mathrm{H}=\frac{1}{2}$, giving 0·5 for the value of the integral K when the argument is $\mathrm{T}\,(=\rho t)=1$. His table gives the values of K to five decimal places only with the argument T, at intervals of 0·01 from $\mathrm{T}=0$ to $\mathrm{T}=3{\cdot}40$ and at intervals of 0·1 from $\mathrm{T}=3{\cdot}4$ to $\mathrm{T}=5$. It was computed from the previous table by direct interpolation, and was also reprinted by DE MORGAN both in his *Theory* and his *Essay*.

Here it may be noted that this second table is so readily derived from a table of the values of H, when these are determined with precision, that there seems little reason for computing it. For if we multiply the arguments in such a table by $1/\rho = 2{\cdot}096\ 716\ 165$, or approximately by $\frac{65}{31}$ or $\frac{629}{300}$, we have at once a table of the values of K, only with arguments at intervals that are inconvenient on account of the fractions. But since the arguments required in practical applications nearly always lie between two consecutive tabular arguments, and interpolation has to be made at any rate, we may as well perform the operation on the values in a table of H as in one of K. This is done by multiplying the argument (T) for K by $\rho = 0{\cdot}476\ 936$, or, approximately by $\frac{31}{65}$, and taking the corresponding value from the table for H. Thus, if for the argument for K we have $\mathrm{T}=3{\cdot}72$, then $3{\cdot}72\times\rho = 1{\cdot}7742 = t$, for which our table gives $\mathrm{H}=0{\cdot}987\ 8960$: and ENCKE's table, by interpolation, for arg. 3·72, gives $\mathrm{K}=0{\cdot}98790$.

But, we might also compute the first part of ENCKE's table from the formula—

$$\begin{aligned}\mathrm{K} = {} & 0{\cdot}538\ 164\ 958\ 101\ 235\mathrm{T} - {\cdot}040\ 805\ 140\ 181\ 145\mathrm{T}^3 + {\cdot}002\ 784\ 561\ 677\ 8354\mathrm{T}^5 \\ & - {\cdot}000\ 150\ 809\ 348\ 77027\mathrm{T}^7 + {\cdot}000\ 006\ 670\ 286\ 943\ 3025\mathrm{T}^9 - {\cdot}000\ 000\ 248\ 189\ 408\mathrm{T}^{11} \\ & + {\cdot}000\ 000\ 007\ 964\ 597\ 724\mathrm{T}^{13} - {\cdot}000\ 000\ 000\ 224\ 304\ 823\mathrm{T}^{15} + {\cdot}000\ 000\ 000\ 005\ 627\ 456\mathrm{T}^{17} \\ & - {\cdot}000\ 000\ 000\ 000\ 127\ 2874\mathrm{T}^{19} + \text{etc.}\end{aligned} \tag{6}$$

This will give values correct to fourteen decimal places, as far as $\mathrm{T}=1$, and seven

* *Berl. Astronom. Jahrbuch* für 1834, S. 269. Mr J. W. L. GLAISHER (*Phil. Mag.* (1871), vol. xlii, p. 434) remarks that, if ENCKE's table were derived from BESSEL's, it must have been "by interpolation from his second table." But he overlooks the fact that BESSEL's Table II. is only a continuation of Table I., giving the logarithmic values of the multiple of the integral by e^{t^2} from $t=1$ to $t=10$, with logarithms of t for argument.

† *Berl. Astron. Jahrb.*, 1834, Ss. 309–312.

terms only will give correct results up to that point to nine places; but at T = 2 (K = ·822 656 449) the whole ten terms will be required to give eight figures correctly. When T^2 consists of only two figures, the computation is easy, if we begin with the term having the highest power of T. For the larger values of T, however, if not for all, it is easier to derive the values of K by interpolation from those of H.

7. It was a suspicion of some errors in the last figures of a few of the values in these two tables in De Morgan's *Essay*, and in some values in Airy's *Theory of Errors of Observations* (1861),* that led me to recompute the table of H. It was begun during a holiday in the hot season of 1862 at an Indian hill sanatorium, where I had very few books, and rather as an amusement to occupy the middle hours of the day, than with any idea of publication.

Commencing on a more extensive scale than Encke's table, in fact computing for intervals of 0·001, the values were worked out to about twelve places, but only nine were preserved, together with first and second differences. To this I added the values † of $\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$, partly as a check on the working, with differences. The work was at that time advanced from $t = 0$ to $t = 1·250$, after which it was entirely laid aside for more than thirty years. The computation of the portion carrying the argument to $t = 3$ is exceedingly laborious, even with the intervals doubled after $t = 1·5$. But the values have been given to fifteen decimal places from computations generally made to three or four figures more, and might have been depended on as accurate even beyond the sixteenth place.

This table, then, as recomputed, besides enabling us to construct Encke's second table of K to seven or more decimal places, affords also the means of reconstructing or verifying and extending Kramp's Table I. (for G) by means of the expression (4). Several important constants also have been computed to a degree of accuracy perhaps beyond any practical requirement.‡

The Formulæ.

8. The formulæ available for computation, as pointed out by Laplace,§ are primarily three,—(8), (10) and (11), with the continued fraction (13), which he supplied to facilitate calculation where the series become very slowly convergent.

(1) In the integral $\int \epsilon^{-t^2} dt$, if we develope ϵ^{-t^2}, we get—

$$\int dt\left(1 - t^2 + \frac{t^4}{1.2} - \frac{t^6}{1.2.3} + \text{etc.}\right) = t - \frac{t^3}{3} + \frac{1}{1.2}\cdot\frac{t^5}{5} - \frac{1}{3!}\cdot\frac{t^7}{7} + \text{etc.,} \tag{7}$$

* *Op. cit.*, pp. 16, 20, 22–24.

1.128 379 167 094 634

† I began by using the value of $\frac{2}{\sqrt{\pi}}$ given in Shortrede's *Logarithmic Tables* (1858), p. 602, viz., ~~1·283 791 670 946 99~~ which is correct only to the tenth place, and therefore could not affect any of the results up to the eleventh place. This was examined later, and the true value of the constant found to be ~~1·283~~ 791 670 955 126. Shortrede's
1.128 379 167 095 512 6
logarithm of $\frac{2}{\sqrt{\pi}}$ is correct. His value of sin 1° is also in error after the tenth decimal.

‡ In the small table given by Airy, *Theory of Errors*, p. 24, six of the constants dependent on ρ are in error in the 5th and 6th places, three of them in the 4th.

§ *Théorie Analytique des Probabilités*, 2e. ed. (1814), p. 103, and *Mécanique Céleste*, liv. x, c. i, sec. 5.

and taking the integral from $t=0$ to $t=t$, we have

$$\int_0^t dt\epsilon^{-t^2}=t-\frac{t^3}{3}+\frac{t^5}{1.2.5}-\frac{t^7}{3!\,7}+\frac{t^9}{4!\,9}-\text{ etc.}=\tfrac{1}{2}\sqrt{\pi}.\mathrm{H}=\tfrac{1}{2}\sqrt{\pi}-\mathrm{G}. \tag{8}$$

That is—

$$\frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2}dt=\frac{2}{\sqrt{\pi}}\left(t-\frac{t^3}{3}+\frac{t^5}{1.2.5}-\frac{t^7}{3!\,7}+\frac{t^9}{4!\,9}-\frac{t^{11}}{5!\,11}+\text{ etc.}\right)=\mathrm{H}. \tag{9}$$

(2) Integration by parts shows at once that—

$$\int t^n dt\epsilon^{-t^2}=\frac{1}{n+1}.t^{n+1}\epsilon^{-t^2}+\frac{2}{n+1}\int t^{n+2}dt\epsilon^{-t^2}.$$

And putting successively $n=0$, $n=2$, $n=4$, $n=6$, etc., we get by repeated substitutions—

$$\int dt\epsilon^{-t^2}=t\epsilon^{-t^2}+2\int t^2dt\epsilon^{-t^2}=\epsilon^{-t^2}\left(t+\frac{2}{3}t^3\right)+\frac{2^2}{3}\int t^4dt\epsilon^{-t^2}=\epsilon^{-t^2}\left(t+\frac{2t^3}{3}+\frac{2^2t^5}{3.5}\right)+\frac{2^3}{3.5}\int t^6dt\epsilon^{-t^2}, \text{ etc.,}$$

which vanishes when $t=0$, and when $t=t$, we have—

$$\int_0^t dt\epsilon^{-t^2}=\epsilon^{-t^2}t\left\{1+\frac{2t^2}{1.3}+\frac{(2t^2)^2}{1.3.5}+\frac{(2t^2)^3}{1.3.5.7}+\text{ etc.}\right\}=\tfrac{1}{2}\sqrt{\pi}.\mathrm{H}. \tag{10}$$

(3) By a process similar to the last we find that—

$$\int t^{-n}dt\epsilon^{-t^2}=-\tfrac{1}{2}t^{-n-1}\epsilon^{-t^2}-\tfrac{1}{2}(n+1)\int t^{-n-2}dt\epsilon^{-t^2}, \text{ etc.}$$

Hence

$$\int dt\epsilon^{-t^2}=\mathrm{C}-\frac{\epsilon^{-t^2}}{2t}\left(1-\frac{1}{2t^2}+\frac{1.3}{(2t^2)^2}-\frac{1.3.5}{(2t^2)^3}+\text{ etc.}\right).$$

Putting $t=\tau$, the constant quantity is eliminated by making the integral vanish, and we have—

$$\int_t^\tau dt\epsilon^{-t^2}=\frac{\epsilon^{-t^2}}{2t}\left(1-\frac{1}{2t^2}+\frac{1.3}{(2t^2)^2}-\frac{1.3.5}{(2t^2)^3}+\text{ etc.}\right)-\frac{\epsilon^{-\tau^2}}{2\tau}\left(1-\frac{1}{2\tau^2}+\frac{1.3}{(2\tau^2)^2}-\text{ etc.}\right).$$

Then putting $\tau=\infty$, we have the series—

$$\int_t^\infty dt\epsilon^{-t^2}=\frac{\epsilon^{-t^2}}{2t}\left(1-\frac{1}{2t^2}+\frac{1.3}{(2t^2)^2}-\frac{1.3.5}{(2t^2)^3}+\text{ etc.}\right)=\mathrm{G}, \tag{11}$$

and

$$\int_0^t dt\epsilon^{-t^2}=\tfrac{1}{2}\sqrt{\pi}-\frac{\epsilon^{-t^2}}{2t}\left(1-\frac{1}{2t^2}+\frac{1.3}{(2t^2)^2}-\frac{1.3.5}{(2t^2)^3}+\text{ etc.}\right)^*=\tfrac{1}{2}\sqrt{\pi}.\mathrm{H} \tag{12}$$

The series (8) and (10) are convergent, but when t exceeds 2, the convergence becomes very slow. The first (8) and third (11) are alternately greater and less than the integral, so that if we add to any number of their terms the half of the following term, the error

* Conf. Hymers' *Integ. Calc.*, pp. 123, 151.

will be less than that half. But the third series (11) is not convergent, the numerators of the successive fractions soon exceeding any value of $2t^2$ that is likely to be used. To meet this case, we have LAPLACE's continued fraction,* into which the series is converted, and which becomes more convergent the higher the value of t. And this can be used for either G or H.

Laplace's Continued Fraction.

9. When $t > 1\cdot5$ it becomes very laborious to compute values of H, and LAPLACE gave the series for $\int_t^\infty dt.\epsilon^{-t^2} = \frac{\epsilon^{-t^2}}{2t}\left\{1-\frac{1}{2t^2}+\frac{1.3}{2^2t^4}-\frac{1.3.5}{2^3t^6}+\text{etc.}\right\}$, the form of a continued fraction, putting $q=\frac{1}{2t^2}$,—

$$\text{G or } \int_t^\infty \epsilon^{-t^2}dt = \frac{\epsilon^{-t^2}}{2t}\cdot\cfrac{1}{1+\cfrac{q}{1+\cfrac{2q}{1+\cfrac{3q}{1+\cfrac{4q}{1+\text{etc.}}}}}}, \tag{13}$$

and this gives a series of common fractions alternately greater and less than the integral. Mr GLAISHER has used this in computing his table of the values of the other function, G, from $t=3$ to $t=4\cdot5$. And for higher values of t the approximation of the successive fractions is increasingly rapid. But at any stage the degree of approximation can be estimated only by reducing two consecutive fractions to decimals. To attain a nearly correct value too, with values of t under 3, the computation of a long series of fractions of the form—

$$\frac{1}{1},\ \frac{1}{1+q},\ \frac{1+2q}{1+3q},\ \frac{1+5q}{1+6q+3q^2},\ \frac{1+9q+8q^2}{1+10q+15q^2},\ \frac{1+14q+33q^2}{1+15q+45q^2+15q^3},\ \text{etc.},$$

becomes tedious. This is obviated to a considerable extent, by determining once for all the coefficients a', b', c', etc., and a, b, c, etc., in the following expressions for the numerator and denominator of the fraction when it involves high powers of q. Thus we get two consecutive fractions of the form—(when n is even)—

$$\left.\begin{aligned} \mathrm{L}_{n-1} &= \frac{1+a'q+b'q^2+c'q^3+\ldots+l'q^{\frac{1}{2}n-1}}{1+aq+bq^2+cq^3+\ldots+lq^{\frac{1}{2}n-1}} \\ \text{and}\qquad \mathrm{L}_n &= \frac{1+(a'+n)q+\ldots+l''q^{\frac{1}{2}n-1}}{1+(a+n)q+\ldots+mq^{\frac{n}{2}}} \end{aligned}\right\} \tag{14}$$

and the numerator and denominator for L_{n+1} are found by multiplying those of L_{n-1} by nq and adding those of L_n.

* See LAPLACE's *Méc. Cél.*, *ut sup.*, and *Theor. Anal. des Probab.*, p. 104; DE MORGAN, "Theory of Probabilities," § 63; and *Diff. and Integ. Calc.*, p. 591.

$$\text{Thus, putting } L_n = \frac{N_n}{D_n}, \qquad L_{n+1} = \frac{nqN_{n-1} + N_n}{nqD_{n-1} + D_n} \tag{15}$$

When n is an even number the fraction L_n is less than the true value, and when odd, it is in excess by a quantity $c < \frac{1}{2}(L_n \sim L_{n+1})$.

The larger t is, the more rapidly the fraction approaches its limit, and consequently a lower value of n in L_n will give a sufficiently close approximation.

The following values of the coefficients of q in L_n can be made to serve in nearly all cases when $t > 1{\cdot}5$:—

$$L_{13} = \frac{1 + 77q + 2070q^2 + 23814q^3 + 114\,765q^4 + 187\,425q^5 + 46080q^6}{1 + 78q + 2145q^2 + 25740q^3 + 135\,135q^4 + 270\,270q^5 + 135\,135q^6} -$$

$$L_{14} = \frac{1 + 90q + 2915q^2 + 42300q^3 + 278\,019q^4 + 729\,330q^5 + 509\,985q^6}{1 + 91q + 3003q^2 + 45045q^3 + 315\,315q^4 + 945\,945q^5 + 945\,945q^6 + 135\,135q^7} +$$

$$L_{15} = \frac{1 + 104q + 3993q^2 + 71280q^3 + 611\,415q^4 + 2336\,040q^5 + 3133\,935q^6 + 645\,120q^7}{1 + 105q + 4095q^2 + 75075q^3 + 675\,675q^4 + 2837\,835q^5 + 4729\,725q^6 + 2027\,025q^7} -$$

$$L_{17} = \frac{1 + 135q + 7007q^2 + 178\,893q^3 + 2386\,395q^4 + 16288\,965q^5 + 51450\,525q^6 + 58437\,855q^7 + 10321\,920q^8}{1 + 136q + 7140q^2 + 185\,640q^3 + 2552\,550q^4 + 18378\,360q^5 + 64324\,260q^6 + 91891\,800q^7 + 34459\,425q^8} -$$

$$L_{18} = \frac{1 + 152q + 9030q^2 + 269\,724q^3 + 4341\,480q^4 + 37469\,520q^5 + 162\,058\,050q^6 + 297\,693\,900q^7 + 151\,335\,135q^8}{1 + 153q + 9180q^2 + 278\,460q^3 + 4594\,590q^4 + 41351\,310q^5 + 192\,972\,780q^6 + 413\,513\,100q^7 + 310\,134\,825q^8 + 34459\,425q^9} +$$

$$L_{20} = \frac{\left.\begin{array}{r}1 + 189q + 14348q^2 + 567\,420q^3 + 12686\,310q^4 + 162\,912\,750q^5 + 1167\,180\,300q^6 + 4302\,906\,300q^7 + 6859\,400\,625q^8 \\ + 3061\,162\,125q^9\end{array}\right\}}{\left.\begin{array}{r}1 + 190q + 14535q^2 + 581\,400q^3 + 13226\,850q^4 + 174\,594\,420q^5 + 1309\,458\,150q^6 + 5237\,832\,600q^7 + 9820\,936\,125q^8 \\ + 6547\,290\,750q^9 + 654\,729\,075q^{10}\end{array}\right\}} +$$

$$L_{21} = \frac{\left.\begin{array}{r}1 + 209q + 17748q^2 + 796\,620q^3 + 20603\,310q^4 + 314\,143\,830q^5 + 2775\,672\,900q^6 + 13408\,094\,700q^7 + 31335\,467\,625q^8 \\ + 27125\,492\,625q^9 + 3716\,891\,200q^{10}\end{array}\right\}}{\left.\begin{array}{r}1 + 210q + 17955q^2 + 813\,960q^4 + 21866\,450q^4 + 333\,316\,620q^5 + 3055\,402\,350q^6 + 15718\,497\,800q^7 + 41247\,931\,725q^8 \\ + 45831\,035\,250q^9 + 13749\,310\,575q^{10}\end{array}\right\}} -$$

$$L_{23} = \frac{\left.\begin{array}{r}1 + 252q + 26315q^2 + 1488\,384q^3 + 50044\,770q^4 + 1038\,829\,160q^5 + 13108\,004\,910q^6 + 98983\,684\,800q^7 + 416\,674\,583\,325q^8 \\ + 860\,553\,193\,500q^9 + 664\,761\,133\,575q^{10} + 81749\,606\,400q^{11}\end{array}\right\}}{\left.\begin{array}{r}1 + 253q + 26565q^2 + 1514\,205q^3 + 51482\,970q^4 + 1081\,142\,370q^5 + 14054\,850\,810q^6 + 110\,430\,970\,650q^7 + 496\,939\,367\,925q^8 \\ + 1159\,525\,191\,825q^9 + 1159\,525\,191\,825q^{10} + 316\,234\,143\,225q^{11}\end{array}\right\}} -$$

$$L_{24} = \frac{\left.\begin{array}{r}1 + 275q + 31605q^2 + 1987\,875q^3 + 75297\,114q^4 + 1781\,769\,150q^5 + 26460\,800\,730q^6 + 241\,511\,019\,750q^7 + 1288\,808\,846\,325q^8 \\ + 3659\,572\,691\,775q^9 + 4601\,737\,965\,825q^{10} + 1645\,756\,410\,375q^{11}\end{array}\right\}}{\left.\begin{array}{r}1 + 276q + 31878q^2 + 2018\,940q^3 + 77224\,455q^4 + 1853\,386\,920q^5 + 28109\,701\,620q^6 + 265\,034\,329\,560q^7 + 1490\,818\,103\,775q^8 \\ + 4638\,100\,767\,300q^9 + 6957\,151\,150\,950q^{10} + 3794\,809\,718\,700q^{11} + 316\,234\,143\,225q^{12}\end{array}\right\}} +$$

The multiplier q being always a proper fraction, we begin by dividing the last coefficient by $2t^2$, add the next preceding and divide again, and so on to the first coefficient of q, adding unity to the last quotient. If, for example, we take $t = 1{\cdot}75$, $q = \frac{8}{49} = \frac{7}{49} + \frac{1}{49}$,—which is easily manipulated—and we find, on dividing down the coefficients in the terms for L_{23}—

$$L_{23} = \frac{1007439{\cdot}089305}{1139733{\cdot}366404} = 0{\cdot}883\,925\,239\,886,$$

and
$$L_{24} = \frac{2535470{\cdot}688789}{2868422{\cdot}115642} = 0{\cdot}883\,925\,233\,655.$$

These agree to the eighth decimal place, the first being too large and the second too small but nearer the true value,—which is $0{\cdot}883\,925\,236\,007\,66$.

For $t = 1{\cdot}75$, the value of ϵ^{-t^2} is $0{\cdot}052\,774\,995\,930\,150\,374\,66$, and since (4)—

$$\mathrm{H}=1-\frac{2}{\sqrt{\pi}}\frac{\epsilon^{-t^2}}{2t}\cdot\mathrm{L}=1-\frac{\epsilon^{-t^2}\mathrm{L}}{t\sqrt{\pi}}, \tag{16}$$

with L_{23} we have $\mathrm{H}=0·986\ 671\ 671\ 161+$
and with L_{24} we have $\mathrm{H}=0·986\ 671\ 671\ 255-$
the true value being, $\mathrm{H}=0·986\ 671\ 671\ 219$.

Hence this degree of approximation, being to the tenth place in decimals, would be practically sufficient for all purposes. And for higher values of t, the results are still more close, and even a lower order of the fraction L would suffice. For $t=3$, L_{22} comes out ·951 818 839 1839 +, which is correct to the last—the 13th—figure.

10. When the values of L_{n-1} and L_n are not sufficiently accordant, either from t being small or n not sufficiently high, we may readily compute L_{n+1}. Then if $\mathrm{L}_{n-1}-\mathrm{L}_n=a$, and $\mathrm{L}_n-\mathrm{L}_{n+1}=b$, we may find a correction $\frac{a^2}{a-b}$, $\frac{ab}{a-b}$, or $\frac{b^2}{a-b}$ (regard being had to the signs of a and b, one of which is always negative), and—

$\mathrm{L}_{n-1}+\dfrac{a^2}{a-b}$, or $\mathrm{L}_n+\dfrac{ab}{a-b}$, or $\mathrm{L}_{n+1}+\dfrac{b^2}{a-b}$,—which will be equal or very nearly so,—will give a closer approximation to the value of L than before. It will be greater or less than the true value, according as L_{n-1} and L_{n+1} are both greater or both less than L_n.*

11. By means of equation (9) we may compute any values of H up to a certain point with considerable facility, but with $t>1$ it becomes rapidly more difficult. We may, however, use it for such values of t as 2, 2·5, and even 3, though the work is lengthy; and for purposes of verification this has been done in the following table. For extreme accuracy the continued fraction is scarcely less laborious, till we reach $t=3$. Up to $t=1·25$ the values were determined for moderate and equal intervals by means of (9), and the intermediate values inserted by interpolation, using the highest order of differences that could by any chance affect the results.

12. We might, however, make use of the method of *quadratures*. For H may be regarded as the area of a curve of which the equation is $y=\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$. Hence the value of $\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$ represents the rate of increment of that area at t; and the area between any two ordinates is the difference of the values of H between the two corresponding values of t. And if the intervals between the ordinates are so small as to enable us to find the area with sufficient accuracy, we may compute values of H,—or rather of the differences of H between two values of t,—with great precision. If, for example, we take the ordinates, given in the first part of the table, from $t=1·160$ to $t=1·170$ inclusive, the area is found by SIMPSON's rule† to be ·002 904 196 086 +, and adding this to the value of H for $t=1·160$ (from the second part of the table), the sum is the value of H when $t=1·170$, viz., 0·902 000 398 966,—which is correct to the last figure.

Or, generally, if V_0, V_1, V_2, . . . V_n, be the values of the successive ordinates whose

* In the example above of $t=1·75$, L_{22} will be 0·883 925 237 509, and $a=-6231$, $b=+3854$, whence the corrections are, -3850, $+2381$, and -1473, respectively, each giving ·883 925 236 036.

† T. SIMPSON's *Mathematical Dissertations* (1743), pp. 109 f. This rule gives a very close approximation. Conf. HYMERS' *Int. Calc.*, p. 181 ; HUTTON's *Mensuration*, p. 374.

distance apart is $\theta = t_1 - t_0$, $\Delta_1 = V_1 - V_0$, $\Delta_1' = V_n - V_{n-1}$, and Δ_2, Δ_2', the first and last of the second differences, and so on; then between V_0 and V_n the area is—

$$\theta\Big\{\Big(\frac{1}{2}V_0 + V_1 + V_2 + \ldots + \tfrac{1}{2}V_n\Big) - \frac{1}{12}(\Delta_1' - \Delta_1) - \frac{1}{24}(\Delta_2' + \Delta_2) - \frac{19}{720}(\Delta_3' - \Delta_3) - \frac{3}{160}(\Delta_4' + \Delta_4)$$
$$- \frac{863}{60\,480}(\Delta_5' - \Delta_5) - \frac{275}{24\,192}(\Delta_6' + \Delta_6) - \frac{33\,953}{3628\,800}(\Delta_7' - \Delta_7) - \frac{8183}{1030\,800}(\Delta_8' + \Delta_8)$$
$$- \frac{3250\,433}{479\,001\,600}(\Delta_9' - \Delta_9)\ldots\Big\}^{*} \qquad (17)$$

of which expression the first three terms will generally be sufficient. Taking the same example, we have—

$$\begin{array}{rlrlrl}
\Delta_1 = & -681\,145 & \Delta_2 = & +994 & \tfrac{1}{2}V_0 + V_1 + \ldots + V_9 + \tfrac{1}{2}V_{10} = & \cdot290\,419\,6916 \\
\Delta_1' = & -\underline{672\,183} & \Delta_2' = & \underline{+997} & -\tfrac{1}{12}(\Delta_1' - \Delta_1) = & -747- \\
\Delta_1' - \Delta_1 = & +8962 & \Delta_2' + \Delta_2 = & +1991 & -\tfrac{1}{24}(\Delta_1' + \Delta_2) = & \underline{-\ 83-} \\
 & & & & \text{Sum,}\ \ . \quad . & \cdot290\,419\,6086+
\end{array}$$

and $\theta = \cdot01$; hence the area is $\cdot002\,904\,196\,086+$, as before.

For a single interval, as between V_0 and V_1, by putting Δ_2^0 for the second difference, derived from V_{-1} and V_1, and Δ_2 the next in succession, derived from V_0 and V_2; Δ_4^0 the fourth difference, in line with V_0, and Δ_4, for the next below, etc., we have the area expressed by—

$$\theta(V_0 + V_1) - \frac{\theta}{24}(\Delta_2^0 + \Delta_2) - \frac{11\theta}{1440}(\Delta_4^0 + \Delta_4) - \frac{191\theta}{120\,960}(\Delta_6^0 + \Delta_6) - \frac{2497\theta}{7257\,600}(\Delta_8^0 + \Delta_8)\ \ldots \qquad (18)$$

Taking the values of V at 1·130, 1·140, . . . and 1·180, we find for 1·160, $\Delta_2^0 = +99\,373$, and $\Delta_2 = +99\,759$, also $\Delta_4^0 = +70$ and $\Delta_4 = +66$. Then—

$$\begin{array}{rr}
\tfrac{1}{2}(\cdot293\,811\,239 + \cdot287\,044\,575) = & \cdot290\,427\,907 \\
-\tfrac{1}{24}(99\,373 + 99\,759) = & -8297+ \\
-\tfrac{11}{1440}(70 + 66) = & \underline{-\quad 1\ \ } \\
\text{Sum, as before, nearly}\ \ . \quad . & \cdot290\,419\,609-.
\end{array}$$

Interpolation.

13. The method of interpolation employed is familiar, but the process may be explained by which the transference is made from the differences found from the computed values, to the differences required for those to be interpolated.† I have not met with it in any text book at my command, and I think the formation of these differences indicates that too much stress may be laid on the common warning that most reliance is to be placed on results which lie nearest the middle of the series of values

* Conf. De Morgan's *Diff. and Integ. Calc.*, pp. 262, 313–318; Woolhouse, *Assurance Mag.*, vol. xi (1864), p. 309. By this method the computation might have been abridged in some portions, had I noticed its advantages earlier.

5 † Mr W. T. B. Woolhouse, in a paper "On Interpolation, Summation, and the Adjustment of Numerical Tables," in *The Assurance Magazine*, 1863–65 (vol. xi, pp. 61–88, 301–332, and vol. xii, pp. 136–176), has developed a formula with necessary tables for interpolating terms in the *middle* interval of a series. The treatment is interesting, and the formulæ are rapidly convergent, but not altogether convenient for computing a lengthy table.

from which the differences used are derived.* It appears that if the intervals between a series of values be sufficiently small and their number so large that the last difference is practically zero, then the results will usually be about equally correct along the whole series,—for the first interpolated value is affected by the last difference.

14. In the computation of the values of any function to be tabulated with equi-different arguments, the two usual formulæ are—

$$V_n = V_0 + an + bn^2 + cn^3 + dn^4 + en^5 + fn^6 + gn^7 + \text{etc.} \tag{19}$$

and $$V_n = V_0 + n\Delta + \frac{n.n-1}{1.2}\Delta_2 + \frac{n.n-1.n-2}{1.2.3}\Delta_3 + \frac{n.n-1.n-2.n-3}{4!}\Delta_4 + \text{etc.} \tag{20}$$

By the first each value has to be computed separately; by the second, if we determine the values of Δ', Δ'_2, Δ'_3, etc., for the intervals to be adopted, the process is reduced to one of continuous addition and subtraction, according as the signs of the differences require. Now the conversion of the one formula into the other is readily effected by means of the numerical values of $\Delta^n 0^m$.† The following table, rearranged and extended to $\Delta^{12}0^{12}$, will suffice for all purposes:—

	0^1	0^2	0^3	0^4	0^5	0^6	0^7	0^8	0^9	0^{10}	0^{11}	0^{12}
Δ^1	1	1	1	1	1	1	1	1	1	1	1	1
Δ^2		2	6	14	30	62	126	254	510	1022	2046	4094
Δ^3			6	36	150	540	1806	5796	18150	55980	171006	519156
Δ^4				24	240	1560	8400	40824	186480	818520	3498000	14676024
Δ^5					120	1800	16800	126000	834120	5103000	29607600	165528000
Δ^6						720	15120	191520	1905120	16435440	129230640	953029440
Δ^7							5040	141120	2328480	29635200	322494480	3162075840
Δ^8								40320	1451520	30240000	479001600	6411968640
Δ^9									362880	16329600	419126400	8083152000
Δ^{10}										3628800	199584000	6187104000
Δ^{11}											39916800	2634508800
Δ^{12}												479001600

* Conf., *e.g.*, DE MORGAN'S *Diff. and Integ. Calc.*, pp. 544, 545; and WOOLHOUSE in *Assur. Mag.*, vol. xi, p. 73, note.

† HERSCHEL, *Examp. of Calculus of Finite Differences*, p. 9. His table extends to $\Delta^{10}0^{10}$ (conf. DE MORGAN, *Diff. and Int. Calc.*, p. 253.) This table is readily computed by the formula—

$$\Delta^{n+1}0^{m+1} = (n+1)(\Delta^n 0^m + \Delta^{n+1}0^m). \tag{21}$$

That is, the sum of the quantities in the two lines for Δ^n and Δ^{n+1}, in the preceding column for 0^m, multiplied by

We have here the coefficients in the following values—

$$\left.\begin{aligned}
\Delta_1 &= a+b+c+d+e+f+g+h+i+k+\text{ etc.}\\
\Delta_2 &= 2b+6c+14d+30e+62f+126g+254h+510i+1022k+\text{ etc.}\\
\Delta_3 &= 6c+36d+150e+540f+1806g+5796h+18150i+55980k,\text{ etc.}\\
\Delta_4 &= 24d+240e+1560f+8400g+40824h+186\,480i+818\,520k,\text{ etc.}\\
\Delta_5 &= 120e+1800f+16800g+126\,000h+834\,120i+5103\,000k,\text{ etc.}\\
\Delta_6 &= 720f+15120g+191\,520h+1905\,120i+16\,435\,440k,\text{ etc.}\\
\Delta_7 &= 5040g+141\,120h+2328\,480i+29\,635\,200k,\text{ etc.}\\
\Delta_8 &= 40320h+1451\,520i+30\,240\,000k,\text{ etc.}\\
\Delta_9 &= 362\,880i+16\,329\,600k,\text{ etc.}\\
\Delta_{10} &= 3628\,800k,\text{ etc.}
\end{aligned}\right\}\quad(22)$$

If we write A, B, C, etc., for the first terms of each value in the above, and reverse the arrangement, we have—

$$\left.\begin{aligned}
\Delta_{10} &= K+\text{ etc.}\\
\Delta_9 &= I+\frac{9}{2}K,\text{ etc.}\\
\Delta_8 &= H+4I+\frac{25}{3}K,\text{ etc.}\\
\Delta_7 &= G+\frac{7}{2}H+\frac{77}{12}I+\frac{49}{6}K,\\
\Delta_6 &= F+3G+\frac{19}{4}H+\frac{21}{4}I+\frac{1087}{240}K,\\
\Delta_5 &= E+\frac{5}{2}F+\frac{10}{3}G+\frac{25}{8}H+\frac{331}{144}I+\frac{45}{32}K,\\
\Delta_4 &= D+2E+\frac{13}{6}F+\frac{5}{3}G+\frac{81}{80}H+\frac{37}{72}I+\frac{6821}{30240}K,\\
\Delta_3 &= C+\frac{3}{2}D+\frac{5}{4}E+\frac{3}{4}F+\frac{43}{120}G+\frac{23}{160}H+\frac{605}{12096}I+\frac{311}{20160}K,\\
\Delta_2 &= B+C+\frac{7}{12}D+\frac{1}{4}E+\frac{31}{360}F+\frac{1}{40}G+\frac{127}{20160}H+\frac{17}{12096}I+\frac{73}{259200}K,\\
\Delta_1 &= A+\frac{1}{2}B+\frac{1}{6}C+\frac{1}{24}D+\frac{E}{5!}+\frac{F}{6!}+\frac{G}{7!}+\frac{H}{8!}+\frac{I}{9!}+\frac{K}{10!}.
\end{aligned}\right\}\quad(23)$$

15. These equations readily give us the values of $A, B, C \ldots K$; and now, if n denote any subdivision of the intervals for which Δ, Δ_2, Δ_3, etc., represent the successive differences, and Δ', Δ'_2, Δ'_3, etc., represent the differences for these smaller intervals in the value of the argument,—then we have—

$$n^{10}\Delta'_{10}=K+\text{ etc.}\qquad n^9\Delta'_9=I+\frac{9K}{2n}+,\text{ etc.}\qquad n^8\Delta'_8=H+\frac{4I}{n}+\frac{25K}{3n^2};\text{ and so on.}\qquad(24)$$

the index of Δ in the second line, gives the value in the 0^{m+1} column: thus $\Delta^3 0^7+\Delta^4 0^7=1806+8400=10206$, and $10206\times 4=40824=\Delta^4 0^8$. The formula is derived from that for $\Delta^n 0^m$ in HERSCHEL's Appendix to LACROIX's *Differ. and Integ. Calculus*, (1816), p. 478.

And, by means of (23) we may thus obtain the values of Δ'_1, Δ'_2, Δ'_3, etc. Or, by transposition, we have

$$\left.\begin{aligned}
n^{10}\Delta'_{10} &= \Delta_{10} = K + \text{etc.} \\
n^9\Delta'_9 &= \Delta_9 - \frac{9(n-1)}{2n}K, \\
n^8\Delta'_8 &= \Delta_8 - \frac{4(n-1)}{n}I - \frac{25(n^2-1)}{3n^2}K, \\
n^7\Delta'_7 &= \Delta_7 - \frac{7(n-1)}{2n}H - \frac{77(n^2-1)}{12n^2}I - \frac{49(n^3-1)}{6n^3}K, \\
n^6\Delta'_6 &= \Delta_6 - \frac{3(n-1)}{n}G - \frac{19(n^2-1)}{4n^2}H - \frac{21(n^3-1)}{4n^3}I - \frac{1087(n^4-1)}{240n^4}K, \text{ etc., etc.}
\end{aligned}\right\} \quad (25)$$

In actual calculation, it is convenient to compute and arrange the quantities in equation (23) thus,—the sum of the quantities in each column being equal to the value of Δ at the top of it:—

Δ_{10}	Δ_9	Δ_8	Δ_7	Δ_6	Δ_5	Δ_4	Δ_3	Δ_2	Δ_1
$K.$	$\frac{9}{2}K.$	$\frac{25}{3}K.$	$\frac{49}{6}K.$	$\frac{1087}{240}K.$	$\frac{45}{32}K.$	$\frac{6821}{3.7.12.120}K.$	$\frac{311}{2.7.12.120}K.$	$\frac{73}{2.9.120.120}K$	$\frac{1}{10!}.K=k.$
	$I.$	$4I.$	$\frac{77}{12}I.$	$\frac{21}{4}I.$	$\frac{331}{144}I.$	$\frac{37}{72}I.$	$\frac{605}{7.12.12.12}I.$	$\frac{17}{7.12.12.12}I.$	$\frac{1}{9!}I=i.$
		$H.$	$\frac{7}{2}H.$	$\frac{19}{4}H.$	$\frac{25}{8}H.$	$\frac{81}{80}H.$	$\frac{23}{160}H.$	$\frac{127}{2.7.12.120}H.$	$\frac{1}{8!}H=h.$
			$G.$	$3G.$	$\frac{10}{3}G.$	$\frac{5}{3}G.$	$\frac{43}{120}G.$	$\frac{1}{40}G.$	$\frac{1}{7!}G=g.$
				$F.$	$\frac{5}{2}F.$	$\frac{13}{6}F.$	$\frac{3}{4}F.$	$\frac{31}{360}F.$	$\frac{1}{6!}F=f.$
					$E.$	$2E.$	$\frac{5}{4}E.$	$\frac{1}{4}E.$	$\frac{1}{5!}E=e.$
						$D.$	$\frac{3}{2}D.$	$\frac{7}{12}D.$	$\frac{1}{4!}D=d.$
							$C.$	$C.$	$\frac{1}{6}C=c.$
								$B.$	$\frac{1}{2}B=b.$
									$A=a.$

After $K=\Delta_{10}$, the values of I, H, G, etc., are successively found by subtracting the sum of the quantities in the proper column from the value of Δ_n above it. If the values (a, b, c, etc.) in the last column are determined with extreme accuracy, they afford a ready means of verification of the whole operation, since

$$V_n - V_0 = an + bn^2 + cn^3 + dn^4 + \dots + kn^{10}.$$

Then, by equation (24), we readily deduce the values of Δ'_1, Δ'_2, etc., from the above by dividing successively upwards each quantity in the column, except the lowest, by n, n^2, n^3, etc., adding the quotients to the value of A, B, or C, etc., and lastly dividing the sum by the coefficient of Δ'. When $n=10$, this can be done by mere inspection. Thus, for example—

$$10^4\Delta'_4 = D+\frac{2E}{10}+\frac{13F}{6.10^2}+\frac{5G}{3.10^3}+\ \dots\ +\frac{6821K}{30240.10^6}.$$

And the quantities in the same horizontal lines may be computed by the fractional coefficients; or, k, i, h, etc., being first found directly from K, I, H, etc., we may use the integral coefficients in eq. (22).

16. Again, if in a series of values of a function, the first differences before and after any value V be Δ_{-1} and Δ; $\Delta^2=\Delta_{-1}-\Delta$; the third differences, before and after Δ^2, be Δ^3_{-1} and Δ^3; $\Delta^4=\Delta^3_{-1}-\Delta^3$; the fifth differences, before and after Δ^4, be $\Delta^5_{-1}-\Delta^5$; and so on,—

$$\left.\begin{aligned}\text{Putting } \Delta^1_0 &= \tfrac{1}{2}(\Delta_{-1}+\Delta) = \Delta-\tfrac{1}{2}\Delta^2,\\ \Delta^3_0 &= \tfrac{1}{2}(\Delta^3_{-1}+\Delta^3) = \Delta^3-\tfrac{1}{2}\Delta^4,\\ \Delta^5_0 &= \tfrac{1}{2}(\Delta^5_{-1}+\Delta^5) = \Delta^5-\tfrac{1}{2}\Delta^6,\\ \Delta^7_0 &= \tfrac{1}{2}(\Delta^7_{-1}+\Delta^7) = \Delta^7-\tfrac{1}{2}\Delta^8,\\ \Delta^9_0 &= \tfrac{1}{2}(\Delta^9_{-1}+\Delta^9) = \Delta^9-\tfrac{1}{2}\Delta^{10}.\end{aligned}\right\} \qquad (26)$$

Then, as before, expressing the values of Δ_0, Δ^2, Δ^3_0, etc., in terms of the coefficients a, b, c . . . in the formula (19), we have—

$$\left.\begin{aligned}\Delta^1_0 &= a+c+e+g+i, & \Delta^2 &= 2(b+d+f+h+k),\\ \Delta^3_0 &= 3\,!\,(c+5e+21g+85i), & \Delta^4 &= 4\,!\,(d+5f+21h+85k),\\ \Delta^5_0 &= 5\,!\,(e+14g+147i), & \Delta^6 &= 6\,!\,(f+14h+147k),\\ \Delta^7_0 &= 7\,!\,(g+30i), & \Delta^8 &= 8\,!\,(h+30k),\\ \Delta^9_0 &= 9\,!\,.i, & \text{and } \Delta^{10} &= 10\,!\,.k.\end{aligned}\right\} \qquad (27)$$

From these we deduce—

$$\left.\begin{aligned}a &= \Delta^1_0-\frac{\Delta^3_0}{3\,!}+\frac{4\Delta^5_0}{5\,!}-\frac{36\Delta^7_0}{7\,!}+\frac{576\Delta^9_0}{9\,!}, & b &= \tfrac{1}{2}\Delta^2-\frac{\Delta^4}{4\,!}+\frac{4\Delta^6}{6\,!}-\frac{36\Delta^8}{8\,!}+\frac{576\Delta^{10}}{10\,!},\\ c &= \frac{\Delta^3_0}{3\,!}-\frac{\Delta^5_0}{4\,!}+\frac{7\Delta^7_0}{6\,!}-\frac{820\Delta^9_0}{9\,!}, & d &= \frac{\Delta^4}{4\,!}-\frac{5\Delta^6}{6\,!}+\frac{49\Delta^8}{8\,!}-\frac{82\Delta^{10}}{9\,!},\\ e &= \frac{\Delta^5_0}{5\,!}-\frac{2\Delta^7_0}{6\,!}+\frac{273\Delta^9_0}{9\,!}, & f &= \frac{\Delta^6}{6\,!}-\frac{14\Delta^8}{8\,!}+\frac{273\Delta^{10}}{10\,!},\\ g &= \frac{\Delta^7_0}{7\,!}-\frac{30\Delta^9_0}{9\,!}, \quad h = \frac{\Delta^8}{8\,!}-\frac{3\Delta^{10}}{9\,!}, & i &= \frac{\Delta^9_0}{9\,!} \quad \text{and } k = \frac{\Delta^{10}}{10\,!}\end{aligned}\right\} (28)$$

Substituting these values in the general form of the function (19), and simplifying, we have—

$$V_n = V + n(\Delta^1_0+\frac{n}{2}\Delta^2)+\frac{n(n^2-1)}{3\,!}(\Delta^3_0+\frac{n}{4}\Delta^4)+\frac{n(n^2-1)(n^2-4)}{5\,!}(\Delta^5_0+\frac{n}{6}\Delta^6)+\text{etc.}^* \qquad (29)$$

* This is only an altered mode of writing the formula given in DE MORGAN'S *Diff. and Integ. Calculus*, p. 546; conf. WOOLHOUSE, *Assur. Mag.*, vol. xi, (1863), p. 68.

and replacing Δ^1_0, Δ^3_0, etc., by the second equivalents from (26) we have finally—

$$V_n = V + n(\Delta + \frac{n-1}{2}\Delta^2) + \frac{n(n^2-1)}{3!}.(\Delta^3 + \frac{n-2}{4}\Delta^4) + \frac{n(n^2-1)(n^2-4)}{5!}.(\Delta^5 + \frac{n-3}{6}\Delta^6)$$
$$+ \frac{n(n^2-1)(n^2-4)(n^2-9)}{7!}.(\Delta^7 + \frac{n-4}{8}\Delta^8) + \text{etc.} \tag{30}$$

Either of these formulæ, which converge rapidly, may be used for interpolating terms in a series of values already found, especially if we form tables of the values of each term for the various coefficients of Δ^2, Δ^3, Δ^4, etc. Thus, to insert values at intervals of 0·1 between V and V_{10}, we have—

$$\left.\begin{aligned}
V_1 &= V + \frac{\Delta}{10} - \cdot045.\Delta^2 - \cdot0165.\Delta^3 + \cdot007\,8375.\Delta^4 + \cdot003\,291\,75.\Delta^5 - \cdot001\,591\,0125.\Delta^6 - \text{etc.}\\
V_2 &= V_1 + \frac{\Delta}{10} - \cdot035.\Delta^2 - \cdot0155.\Delta^3 + \cdot006\,5625.\Delta^4 + \cdot003\,044\,25.\Delta^5 - \cdot001\,365\,7875.\Delta^6 - \text{etc.}\\
V_3 &= V_2 + \frac{\Delta}{10} - \cdot025.\Delta^2 - \cdot0135\,\Delta^3 + \cdot004\,9375.\Delta^4 + \cdot002\,559\,25\,\Delta^5 - \cdot001\,046\,06\,25.\Delta^6 - \text{etc.}\\
V_4 &= V_3 + \frac{\Delta}{10} - \cdot015.\Delta^2 - \cdot0105\,\Delta^3 + \cdot003\,0625.\Delta^4 + \cdot001\,856\,75.\Delta^5 - \cdot000\,656\,3375.\Delta^6 - \text{etc.}\\
V_5 &= V_4 + \frac{\Delta}{10} - \cdot005.\Delta^2 - \cdot0065.\Delta^3 + \cdot001\,0375.\Delta^4 + \cdot000\,966\,75.\Delta^5 - \cdot000\,222\,9125.\Delta^6 - \text{etc.}\\
V_6 &= V_5 + \frac{\Delta}{10} + \cdot005.\Delta^2 - \cdot0015.\Delta^3 - \cdot001\,0375.\Delta^4 - \cdot000\,070\,75.\Delta^5 + \cdot000\,222\,9125.\Delta^6 + \text{etc.}\\
V_7 &= V_6 + \frac{\Delta}{10} + \cdot015.\Delta^2 + \cdot0045.\Delta^3 - \cdot003\,0625\,\Delta^4 - \cdot001\,205\,75.\Delta^5 + \cdot000\,656\,3375.\Delta^6 + \text{etc.}\\
V_8 &= V_7 + \frac{\Delta}{10} + \cdot025\,\Delta^2 + \cdot0115.\Delta^3 - \cdot004\,9375.\Delta^4 - \cdot002\,378\,25.\Delta^5 + \cdot001\,046\,0625.\Delta^6 + \text{etc.}\\
V_9 &= V_8 + \frac{\Delta}{10} + \cdot035.\Delta^2 + \cdot0195.\Delta^3 - \cdot006\,5625.\Delta^4 - \cdot003\,518\,25.\Delta^5 + \cdot001\,365\,7875.\Delta^6 + \text{etc.}\\
V_{10} &= V_9 + \frac{\Delta}{10} + \cdot045.\Delta^2 + \cdot0285.\Delta^3 - \cdot007\,8375.\Delta^4 - \cdot004\,545\,75.\Delta^5 + \cdot001\,591\,0125.\Delta^6 + \text{etc.}
\end{aligned}\right\} \tag{31}$$

If the interval n be $\frac{1}{5}$, we have—

$$\left.\begin{aligned}
V_1 &= V + 0\cdot2\Delta - \cdot08.\Delta^2 - \cdot032.\Delta^3 + \cdot0144\,\Delta^4 + \cdot006336.\Delta^5 - \cdot0029568.\Delta^6 - \text{etc.}\\
V_2 &= V_1 + 0\cdot2\Delta - \cdot04.\Delta^2 - \cdot024.\Delta^3 + \cdot008.\,\Delta^4 + \cdot004416\,\Delta^5 - \cdot0017024.\Delta^6 - \text{etc.}\\
V_3 &= V_2 + 0\cdot2\Delta \qquad - \cdot008.\Delta^3 \qquad + \cdot000896.\Delta^5 \qquad - \text{etc.}\\
V_4 &= V_3 + 0\cdot2\Delta + \cdot04.\Delta^2 + \cdot016.\Delta^3 - \cdot008.\,\Delta^4 - \cdot003584.\Delta^5 + \cdot0017024\,\Delta^6 + \text{etc.}\\
V_5 &= V_4 + 0\cdot2\Delta + \cdot08.\Delta^2 + \cdot048.\Delta^3 - \cdot0144.\Delta^4 - \cdot008064\,\Delta^5 + \cdot0029568.\Delta^6 + \text{etc.}
\end{aligned}\right\} \tag{32}$$

The series converges so rapidly that it is seldom necessary to go beyond the fourth or fifth differences, and the last result in each case is a check on the accuracy of the work. But, as it requires fresh arrangements for each short series of interpolated values, it is not so satisfactory for computing a lengthy table as the method above explained, though a larger number of differences is required to compensate for the more rapid convergence. For isolated values, however (30), is most convenient. We may proceed by successively correcting the differences in a retrograde order, correcting the highest employed, if necessary, to its mean value, by adding half the next above it. Thus, if five orders of difference are to be used, make $\Delta^5_c = \Delta^5 + \frac{1}{2}\Delta^6$. Then—

$$\Delta_c^4 = \Delta^4 + \frac{2+n}{5}\Delta_c^5,\quad \Delta_c^3 = \Delta^3 - \frac{2-n}{4}\Delta_c^4,\quad \Delta_c^2 = \Delta^2 + \frac{1+n}{3}\Delta_c^3,\quad \Delta_c = \Delta - \frac{1-n}{2}\Delta_c^2,\ \text{and } V_n = V + n\Delta_c$$

To bisect an interval, $n = \frac{1}{2}$, and—

$$V_{\frac{1}{2}} = V + \frac{\Delta}{2} - \frac{\Delta^2}{8} - \frac{\Delta^3}{16} + \frac{3\Delta^4}{128} + \frac{3\Delta^5}{256} - \frac{5\Delta^6}{1024} - \frac{5\Delta^7}{2048} + \frac{35\Delta^8}{32768} + \text{etc.} \tag{33}$$

Or, $\Delta_c^4 = \Delta^4 + \frac{1}{2}(\Delta^5 + \frac{1}{2}\Delta^6)$, $\Delta_c^3 = \Delta^3 - \frac{3}{8}\Delta_c^4$, $\Delta_c^2 = \Delta^2 + \frac{1}{2}\Delta_c^3$, $\Delta_c = \Delta - \frac{1}{4}\Delta_c^2$, and $V_{\frac{1}{2}} = V + \frac{1}{2}\Delta_c$. (34)

Thus if it be required to find the value of H corresponding to $t = 1{\cdot}575$, we take the differences following and on line with 1·574 in the table, and proceed thus :—

Δ^4	Δ^3	Δ^2	Δ	H
−18656	+5985 292	−1 192 978 427	+188 870 390 940	·973 983 952 882 675
$-13 = \frac{1}{2}\Delta^5$	= +7 001	= +2 996 146	= + 297 470 570	= +94 583 930 755
$-18669 \times -\frac{3}{8}$	$+5992\ 293 \times \frac{1}{2}$	$-1\ 189\ 882\ 281 \times -\frac{1}{4}$	$+189\ 167\ 861\ 510 \times \frac{1}{2}$	·974 078 536 813 430.

This value, H = ·974 078 536 813 430, is correct to the last figure, and $\frac{1}{2}\Delta^5 = -13$, is so small that it might have been neglected without affecting the result.

After determining the values of H for moderate intervals, the differences for the smaller intervals of ·001 or ·002 were determined by means of the formulæ (22) to (25), and the table thus filled up throughout.

The Difference Formula.

17. The difficulty of computation, due to the slowness of convergence of the series for values of t above 1·0, led KRAMP, who computed the table so often reprinted, to adopt a difference-formula* obtained from the general series by means of TAYLOR's theorem, viz.—

$$\Delta\int_t^\infty \epsilon^{-t^2}dt = -r\epsilon^{-t^2}\left(1 - rt + \frac{2t^2-1}{3}r^2 - \frac{2t^3-3t}{6}r^3 + \text{etc.}\right), \tag{35}$$

where $\Delta t = r = 0{\cdot}01$. This implies the separate computation of the values of the differences for each entry in the table. When r is small, three terms of the series may be sufficient, and M. KRAMP says he used no more. Mr J. W. L. GLAISHER, in computing the values of the same function from $t = 3$ to $t = 4{\cdot}50$, tells us that he computed separate tables of log. ϵ^{-t^2} and of log. $\left(r - tr^2 + \frac{2t^2-1}{3}r^3 - \frac{2t^3-3t}{6}r^4\right)$, and then built up his table by the successive differences.† This requires for his table about a hundred and fifty computations of the values of (35), and an error in one would have been perpetuated

* *Analyse des Refractions astronomiques et terrestres* (Strasbourg, 1799), p. 135.

† *Philos. Mag.*, xlii, (1871), p. 434. Conf. DE MORGAN, *ut cit.*, § 117. Mr GLAISHER remarks (p. 432) that "KRAMP does not state what value he started from in applying the differences, or what means of verification he adopted. In all cases where a table is constructed by means of differences, the last value should be calculated independently, and then the agreement of the two values would verify all the preceding portion of the table." And he adds that KRAMP's value for $t = 3$ is in error in the tenth and eleventh figures, so that probably a portion of his table is incorrect in the last two figures (see § 4 above).

through the rest, if he had not checked his work by means of LAPLACE's continued fraction.

18. But the formula may be applied with great effect in this way: r may be taken as negative as well as positive, so that from a value H, corresponding to t, we can derive both the values at $t-r$ and $t+r$; and by developing the formula more fully, we may use it with much larger values than $r=0{\cdot}01$. Putting $x=-2t$, the general term is—

$$\frac{1}{n+1}\left\{\frac{x^n}{n!}-\frac{x^{n-2}}{(n-2)!}+\frac{x^{n-4}}{1.2.(n-4)!}-\frac{x^{n-6}}{3!(n-6)!}+\frac{x^{n-8}}{4!(n-8)!}-\text{etc.}\right\}$$

Expanding and adapting to the integral $\frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2}dt$,—

$$\left.\begin{aligned}\Delta\mathrm{H}=\frac{2\epsilon^{-t^2}}{\sqrt{\pi}}r\Big\{&1-rt+\frac{2t^2-1}{3}r^2-\frac{2t^3-3t}{6}r^3+\frac{4t^4-12t^2+3}{2.3.5}r^4-\frac{4t^5-20t^3+15t}{3.5.6}r^5\\&+\frac{8t^6-60t^4+90t^2-15}{3.5.7.3\,!}r^6-\frac{8t^7-84t^5+210t^3-105t}{3.5.7.4\,!}r^7+\frac{16t^8-224t^6+840t^4-840t^2+105}{3.5.7.9.4\,!}r^8\\&-\frac{16t^9-288t^7+1512t^5-2520t^3+945t}{3.5.7.9.5\,!}r^9+\frac{32t^{10}-720t^8+5040t^6-12600t^4+9450t^2-945}{3.5.7.9.11.5\,!}r^{10}\\&-\frac{32t^{11}-880t^9+7920t^7-27720t^5+34650t^3-10395t}{3.5.7.9.11.6\,!}r^{11}\\&+\frac{64t^{12}-2112t^{10}+23760t^8-110\,880t^6+207\,900t^4-124\,740t^2+10395}{3.5.7.9.11.13.6\,!}r^{12}\\&-\frac{64t^{13}-2496t^{11}+34320t^9-205\,920t^7+540\,540t^5-540\,540t^3+135\,135t}{3.5.7.9.11.13.7\,!}r^{13}+\text{etc.}\Big\}.^*\end{aligned}\right\}\quad(36)$$

For any portion of the table then, say from $t=1{\cdot}9$ to $t=3$, we may compute the coefficients of the powers of r for t at the values $2{\cdot}0$, $2{\cdot}2$, $2{\cdot}4$, $2{\cdot}6$, $2{\cdot}8$, and 3; and by means of the first we find the differences from $t=1{\cdot}90$ to $t=2{\cdot}10$, by the second series from $t=2{\cdot}10$ to $2{\cdot}30$, and so on. If, also, we know the values for $t=2$ and $t=3$ (which I have computed separately, both by the general series and by LAPLACE's fraction), we can fill up the table,—first, for all values of t differing by $0{\cdot}01$; and, secondly, by forming from these values the differences in the series $\mathrm{H}_0+n\Delta'+\frac{n.\overline{n-1}}{2}\Delta'_2+$ etc., for 1/5, 1/10, or any other subdivision of the interval, we may complete the table from $t=1{\cdot}900$ to $t=3{\cdot}100$. This sufficiently explains the method of computation for the portion of the table beyond $t=1{\cdot}000$.

19. Since the computation of these coefficients of the powers of r is also required for the other branch of the integral—G, they may be preserved here.

For $t=1$, $\epsilon^{-1}=0{\cdot}367\,879\,441\,171\,442\,321\,595\,524$,

$$\begin{aligned}\Delta\mathrm{G}=-r\epsilon^{-t^2}\Big(&1-r+\tfrac{1}{3}r^2+\tfrac{1}{6}r^3-\tfrac{1}{6}r^4+\tfrac{1}{90}r^5+{\cdot}036\,507\dot{9}r^6-{\cdot}011\,507\,93\dot{6}r^7-{\cdot}004\,541\,446\,208\,112\,875r^8\\&+{\cdot}002\,954\,144\,620\,811\,287r^9+{\cdot}000\,206\,028\,539\,362r^{10}-{\cdot}000\,481\,935\,7597r^{11}\\&+{\cdot}000\,045\,088\,6562r^{12}+{\cdot}000\,057\,110\,732r^{13},\text{ etc.}\Big).\end{aligned}$$

Also $\frac{2}{\sqrt{\pi}}\epsilon^{-1}=0{\cdot}415\,107\,497\,420\,594\,703\,340\,268=\mathrm{E}$, and $\mathrm{H}=0{\cdot}842\,700\,792\,949\,714\,869\,34$.

* If we make $t=0$ in this series, r then becomes t, and we have the series in (9) from which it is derived.

And, for our purpose, multiplying the above coefficients by $\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$, we have—

$$\begin{aligned}\Delta H = Er &- 0{\cdot}415\,107\,497\,420\,594\,703\,340\,27r^2 + 0{\cdot}138\,369\,165\,806\,864\,901\,113r^3\\ &+ 0{\cdot}069\,184\,582\,903\,432\,450\,557r^4 - 0{\cdot}069\,184\,582\,903\,432\,4506r^5\\ &+ 0{\cdot}004\,612\,305\,526\,895\,496\,70r^6 + 0{\cdot}015\,154\,718\,159\,799\,49r^7 - 0{\cdot}004\,777\,030\,724\,284\,62r^8\\ &- 0{\cdot}001\,885\,188\,370\,120r^9 + 0{\cdot}001\,226\,287\,580\,563r^{10} + 0{\cdot}000\,085\,523\,9914r^{11}\\ &- 0{\cdot}000\,200\,055\,147r^{12} + 0{\cdot}000\,018\,716\,6392r^{13} + 0{\cdot}000\,023\,707\,93r^{14}.\end{aligned}$$

These values would be sufficient to compute to seventeen or eighteen places all values from $t=0{\cdot}90$ to $t=1{\cdot}10$, making r negative for values below $1{\cdot}0$; and, taken to r^8, they would give accurate results to ten or eleven decimal places.

For $t=1{\cdot}1$, $\epsilon^{-t^2}=0{\cdot}298\,197\,279\,429\,887\,378\,618\,226$.

$$\begin{aligned}\Delta G = -r\epsilon^{-t^2}\{1 &- 1{\cdot}1r + {\cdot}47\dot{3}r^2 + {\cdot}106\dot{3}r^3 - {\cdot}188\,78\dot{6}r^4 + {\cdot}040\,866\dot{2}r^5 + {\cdot}032\,105\,5365\tfrac{5}{63}r^6\\ &- {\cdot}017\,586\,07\tfrac{1}{63}r^7 - {\cdot}001\,943\,926\,059\,964\,7266r^8 + {\cdot}003\,554\,076\,205\,855\,38r^9\\ &- {\cdot}000\,392\,718\,249\,540\,48r^{10} - {\cdot}000\,466\,498\,049\,0775r^{11} + {\cdot}000\,134\,329\,166\,57r^{12}\\ &+ {\cdot}000\,040\,407\,3572r^{13}, \text{ etc.}\}.\end{aligned}$$

and $\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}=0{\cdot}336\,479\,597\,793\,244\,144\,101\,453.$

For $t=1{\cdot}25=\frac{5}{4}$. $\epsilon^{-t^2}=0{\cdot}209\,610\,951\,665\,850\,449\,333\,13$;

$$\begin{aligned}\Delta G = -r\epsilon^{-t^2}\{1 &- 1{\cdot}25r + {\cdot}70\dot{8}\dot{3}r^2 - {\cdot}026\,41\dot{6}r^3 - {\cdot}199\,4791\dot{6}r^4 + {\cdot}090\,060\,763\dot{8}r^5\\ &+ {\cdot}015\,330\,481\,1\dot{5}0\,793\dot{6}r^6 - {\cdot}024\,089\,510\,478\,670\,635r^7 + {\cdot}003\,710\,603\,798\,0875r^8\\ &+ {\cdot}003\,354\,928\,691\,130\,68r^9 - {\cdot}001\,369\,673\,505\,853r^{10} - {\cdot}000\,222\,973\,881\,906r^{11}\\ &+ {\cdot}000\,236\,037\,9076r^{12} - {\cdot}000\,012\,746\,614r^{13}, \text{ etc.}\}.\end{aligned}$$

$\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}={\cdot}236\,521\,122\,447\,290\,787\,220\,015=E$; $H=0{\cdot}922\,900\,128\,256\,458\,230\,14$;

$$\begin{aligned}\text{and } \Delta H = Er &- {\cdot}295\,651\,403\,059\,113\,348\,402\,50r^2 + {\cdot}167\,535\,795\,066\,830\,974\,28r^3\\ &- {\cdot}006\,159\,404\,230\,398\,197\,58r^4 - {\cdot}047\,181\,036\,404\,850\,1935r^5\\ &+ {\cdot}021\,301\,272\,963\,460\,433r^6 + {\cdot}003\,625\,982\,609\,442\,75r^7\\ &- {\cdot}005\,697\,678\,057\,620\,95r^8 + {\cdot}000\,877\,636\,175\,281r^9 + {\cdot}000\,793\,511\,499\,76r^{10}\\ &- {\cdot}000\,323\,956\,7150r^{11} - {\cdot}000\,052\,738\,033r^{12} + {\cdot}000\,055\,827\,95r^{13} - {\cdot}000\,003\,0148r^{14}\end{aligned}$$

$t=1{\cdot}4$: $\epsilon^{-t^2}=0{\cdot}140\,858\,420\,921\,044\,996\,147\,971$;

$$\begin{aligned}\Delta G = -r\epsilon^{-t^2}\{1 &- 1{\cdot}4r + {\cdot}97\dot{3}r^2 - {\cdot}214\dot{6}r^3 - {\cdot}171\,78\dot{6}r^4 + {\cdot}137\,411\dot{5}r^5 - {\cdot}014\,063\,0\dot{3}4\,920\dot{6}r^6\\ &- {\cdot}024\,523\,2\dot{7}1r^7 + {\cdot}010\,363\,941\,013\,58r^8 + {\cdot}001\,457\,789\,123\,950\,06r^9\\ &- {\cdot}002\,066\,991\,235\,5915r^{10} + {\cdot}000\,261\,420\,814\,979r^{11} + {\cdot}000\,235\,192\,7423r^{12}\\ &- {\cdot}000\,081\,536\,3055r^{13}\}\end{aligned}$$

$\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}=0{\cdot}158\,941\,707\,677\,277\,875\,860\,084=E$; $H=0{\cdot}952\,285\,119\,762\,648\,810\,5165$;

$$\begin{aligned}\Delta H = Er &- {\cdot}222\,518\,390\,748\,189\,026\,2041r^2 + {\cdot}154\,703\,262\,139\,217\,132\,504r^3\\ &- {\cdot}034\,119\,486\,581\,388\,984\,02r^4 - {\cdot}027\,304\,066\,156\,187\,3087r^5\\ &+ {\cdot}021\,840\,427\,294\,591\,140r^6 - {\cdot}002\,235\,202\,785\,41091r^7 - {\cdot}003\,897\,770\,588\,2329r^8\\ &+ {\cdot}001\,647\,262\,502\,391r^9 + {\cdot}000\,231\,703\,492\,79r^{10} - {\cdot}000\,328\,531\,1167r^{11}\\ &+ {\cdot}000\,041\,550\,671r^{12} + {\cdot}000\,037\,381\,94r^{13} - {\cdot}000\,012\,9579r^{14}\end{aligned}$$

For $t=1{\cdot}5$ $\epsilon^{-t^2}={\cdot}105\,399\,224\,561\,864\,336\,783\,218$;

$$\begin{aligned}\Delta G = -r\epsilon^{-t^2}\{1 &- 1{\cdot}5r + 1{\cdot}1\dot{6}r^2 - 0{\cdot}375r^3 - {\cdot}125r^4 + 0{\cdot}1625r^5 - 0{\cdot}039\,8\dot{8}0\,952\dot{3}r^6 - 0{\cdot}019\,866\,0\dot{7}1\,428\dot{5}r^7\\ &+ 0{\cdot}014\,376\,653\,439r^8 - 0{\cdot}000\,78125r^9 - 0{\cdot}002\,139\,475\,108r^{10} + 0{\cdot}000\,653\,239\,989r^{11}\\ &+ 0{\cdot}000\,150\,973\,16r^{12} - 0{\cdot}000\,473\,971\,71r^{13}\}.\end{aligned}$$

$$\frac{2}{\sqrt{\pi}}e^{-t^2}=0{\cdot}118\,930\,289\,223\,629\,371\,531\,02=\mathrm{E};\quad \mathrm{H}=0{\cdot}966\,105\,146\,475\,310\,727\,067;$$

and
$$\begin{aligned}\Delta\mathrm{H}=\mathrm{E}r&-0{\cdot}178\,395\,433\,835\,444\,057\,297r^2+0{\cdot}138\,752\,004\,094\,234\,2668r^3\\&-0{\cdot}044\,598\,858\,458\,861\,0143r^4-0{\cdot}014\,866\,286\,152\,953\,671r^5\\&+0{\cdot}019\,326\,171\,998\,839\,77r^6-0{\cdot}004\,743\,053\,201\,180\,46r^7-0{\cdot}002\,362\,677\,620\,737r^8\\&+0{\cdot}001\,709\,819\,551\,58r^9-0{\cdot}000\,092\,914\,288\,46r^{10}-0{\cdot}000\,254\,448\,393r^{11}\\&+0{\cdot}000\,077\,690\,02r^{12}+0{\cdot}000\,017\,955\,28r^{13}-0{\cdot}000\,014\,092r^{14}.\end{aligned}$$

For $t=1{\cdot}6$, $e^{-t^2}={\cdot}077\,304\,740\,443\,299\,745\,990\,466$;

$$\begin{aligned}\Delta\mathrm{G}=-re^{-t^2}\{1&-1{\cdot}6r+1{\cdot}37\dot{3}r^2-0{\cdot}565\dot{3}r^3-0{\cdot}050\,18\dot{6}r^4+0{\cdot}177\,521\dot{7}r^5-0{\cdot}069\,203\,\dot{6}06\,349\dot{2}r^6\\&-0{\cdot}010\,358\,93\dot{8}\,412\,6\dot{9}r^7+0{\cdot}017\,139\,434\,892\,416r^8-0{\cdot}003\,643\,030\,1144r^9\\&-0{\cdot}001\,744\,844\,218r^{10}+0{\cdot}001\,017\,260\,52r^{11}-0{\cdot}000\,004\,336r^{12}\\&-0{\cdot}000\,133\,153\,85r^{13}+\}.\end{aligned}$$

$$\frac{2}{\sqrt{\pi}}e^{-t^2}=0{\cdot}087\,229\,058\,633\,945\,352\,846\,147=\mathrm{E};\quad \mathrm{H}=0{\cdot}976\,348\,383\,344\,644\,007\,77$$

$$\begin{aligned}\Delta\mathrm{H}=\mathrm{E}r&-0{\cdot}139\,566\,493\,814\,312\,564\,553\,836r^2+0{\cdot}119\,794\,573\,857\,284\,951\,242r^3\\&-0{\cdot}049\,313\,494\,481\,057\,106\,14r^4-0{\cdot}004\,377\,735\,689\,308\,937\,44r^5\\&+0{\cdot}015\,485\,057\,562\,579\,995r^6-0{\cdot}006\,036\,565\,435\,915\,39r^7\\&-0{\cdot}000\,903\,600\,446\,1867r^8+0{\cdot}001\,495\,056\,771\,183r^9-0{\cdot}000\,317\,778\,087\,45r^{10}\\&-0{\cdot}000\,152\,201\,1186r^{11}+0{\cdot}000\,088\,734\,678r^{12}-0{\cdot}000\,000\,378\,2405r^{13}\\&-0{\cdot}000\,011\,614\,885r^{14}.\end{aligned}$$

For $t=1{\cdot}8$, $e^{-t^2}={\cdot}039\,163\,895\,098\,987\,073\,739\,770\,994$

$$\begin{aligned}\Delta\mathrm{G}=-re^{-t^2}\{1&-1{\cdot}8r+1{\cdot}82\dot{6}r^2-1{\cdot}044r^3+0{\cdot}203\,68r^4+0{\cdot}156\,192r^5-0{\cdot}128\,822\,\dot{5}52\,380\dot{9}r^6\\&+0{\cdot}024\,500\,434\dot{3}r^7+0{\cdot}015\,248\,655\,\dot{9}15\,34\dot{3}r^8-0{\cdot}009\,845\,148\,89\dot{1}r^9\\&+0{\cdot}000\,726\,814\,123\,775r^{10}+0{\cdot}001\,273\,644\,989r^{11}-0{\cdot}000\,455\,201\,117r^{12}\\&-0{\cdot}000\,509\,014\,6956r^{13}+\text{etc.}\}.\end{aligned}$$

$$\frac{2}{\sqrt{\pi}}e^{-t^2}=0{\cdot}044\,191\,723\,332\,011\,061\,234\,953\,87=\mathrm{E};\quad \mathrm{H}=0{\cdot}989\,090\,501\,635\,730\,714\,19;$$

$$\begin{aligned}\Delta\mathrm{H}=\mathrm{E}r&-0{\cdot}079\,545\,101\,997\,619\,910\,222\,917r^2+0{\cdot}080\,723\,547\,953\,140\,205\,189r^3\\&-0{\cdot}046\,136\,159\,158\,619\,547\,93r^4+0{\cdot}009\,000\,970\,208\,264\,013r^5\\&+0{\cdot}006\,902\,393\,650\,673\,472r^6-0{\cdot}005\,692\,890\,593\,742\,55r^7+0{\cdot}001\,082\,716\,413\,4684r^8\\&+0{\cdot}000\,673\,864\,383\,396r^9-0{\cdot}000\,435\,074\,095\,97r^{10}+0{\cdot}000\,032\,119\,1687r^{11}\\&+0{\cdot}000\,056\,284\,567r^{12}-0{\cdot}000\,020\,116\,12r^{13}-0{\cdot}000\,002\,2494r^{14}+\text{etc.}\end{aligned}$$

For $t=2$, $e^{-t^2}=0{\cdot}018\,315\,638\,888\,734\,180\,293\,718$,

$$\begin{aligned}\Delta\mathrm{G}=-re^{-t^2}\{1&-2r+2{\cdot}\dot{3}r^2-1{\cdot}\dot{6}r^3+0{\cdot}6\dot{3}r^4+0{\cdot}0\dot{2}r^5-0{\cdot}\dot{1}63\,492\dot{0}r^6+0{\cdot}076\,\dot{9}84\,12\dot{6}r^7\\&-0{\cdot}002\,425\,044\,091\,71r^8-0{\cdot}01\dot{2}\,716\,049\,3\dot{8}r^9+0{\cdot}005\,020\,843\,354r^{10}\\&+0{\cdot}000\,253\,059\,6975r^{11}-0{\cdot}000\,785\,932\,1748r^{12}+0{\cdot}000\,191\,191\,54r^{13}-\text{etc.}\}.\end{aligned}$$

$$\frac{2}{\sqrt{\pi}}e^{-t^2}=0{\cdot}020\,666\,985\,354\,092\,053\,857\,069=\mathrm{E};\quad \mathrm{H}=0{\cdot}995\,322\,265\,018\,952\,734\,1517;$$

$$\begin{aligned}\Delta\mathrm{H}=\mathrm{E}r&-0{\cdot}041\,333\,970\,708\,184\,107\,714\,14r^2+0{\cdot}048\,222\,965\,826\,214\,792\,3332r^3\\&-0{\cdot}034\,444\,975\,590\,153\,423\,095r^4+0{\cdot}013\,089\,090\,724\,258\,300\,78r^5\\&+0{\cdot}000\,459\,266\,341\,202\,0456r^6-0{\cdot}003\,378\,888\,081\,700\,764r^7\\&+0{\cdot}001\,591\,029\,824\,878\,52r^8-0{\cdot}000\,050\,118\,350\,7264r^9-0{\cdot}000\,262\,802\,406\,3545r^{10}\\&+0{\cdot}000\,103\,765\,696\,06r^{11}+0{\cdot}000\,005\,229\,9811r^{12}-0{\cdot}000\,016\,242\,849r^{13}\\&+0{\cdot}000\,003\,95114r^{14}-\text{etc.}\end{aligned}$$

For $t=2{\cdot}2$, $e^{-t^2}={\cdot}007\,907\,054\,051\,593\,440\,493\,635\,645$,

$$\Delta G = -r\epsilon^{-t^2}\{1 - 2{\cdot}2r + 2{\cdot}89\dot{3}r^2 - 2{\cdot}449\dot{3}r^3 + 1{\cdot}287\,41\dot{3}r^4 - {\cdot}290\,947\dot{5}r^5 - {\cdot}123\,645\,66\dot{3}\,492\dot{0}r^6$$
$$+ {\cdot}130\,351\,01\dot{9}\,682\,5\dot{3}r^7 - {\cdot}039\,684\,952\,832\,4515r^8 - {\cdot}005\,712\,135\,363\,95r^9$$
$$+ {\cdot}008\,778\,755\,518\,163r^{10} - {\cdot}002\,353\,401\,968\,184r^{11} - {\cdot}000\,441\,493\,573\,588r^{12}$$
$$+ {\cdot}000\,449\,093\,84r^{13} + \}.$$

$$\frac{2}{\sqrt{\pi}}\epsilon^{-t^2} = {\cdot}008\,922\,155\,064\,916\,204\,491\,2763 = E;$$

$$\Delta H = Er - {\cdot}019\,628\,741\,142\,815\,649\,8808r^2 + {\cdot}025\,814\,768\,654\,490\,8850r^3$$
$$- {\cdot}021\,853\,331\,805\,668\,0902r^4 + {\cdot}011\,486\,501\,392\,640\,65r^5 - {\cdot}002\,595\,879\,206\,4449r^6$$
$$- {\cdot}001\,103\,185\,782\,780r^7 + {\cdot}001\,163\,012\,010\,47r^8 - {\cdot}000\,354\,075\,302\,915r^9$$
$$- {\cdot}000\,050\,964\,5574r^{10} + {\cdot}000\,078\,325\,418r^{11} - {\cdot}000\,020\,997\,4173r^{12}$$
$$- {\cdot}000\,003\,939\,07r^{13} + {\cdot}000\,004\,0068r^{14} + \text{etc.}$$

For $t = 2{\cdot}4$, $\epsilon^{-t^2} = 0{\cdot}003\,151\,111\,598\,444\,440\,557\,819\,11$,

$$\Delta G = -r\epsilon^{-t^2}\{1 - 2{\cdot}4r + 3{\cdot}50\dot{6}r^2 - 3{\cdot}408r^3 + 2{\cdot}21968r^4 - {\cdot}866\,944r^5 + {\cdot}065\,980\,6\dot{4}7\,619\dot{0}r^6$$
$$+ {\cdot}146\,185\,32\tfrac{4}{7}r^7 - {\cdot}090\,795\,077\,\dot{4}17\,98\dot{9}r^8 + {\cdot}017\,593\,1348\tfrac{1}{7}r^9$$
$$+ {\cdot}007\,180\,372\,023\,41r^{10} - {\cdot}005\,537\,775\,295\,94r^{11} + {\cdot}001\,032\,100\,464\,94r^{12}$$
$$+ {\cdot}000\,376\,393\,066r^{13}\}.$$

$$\frac{2}{\sqrt{\pi}}\epsilon^{-t^2} = 0{\cdot}003\,555\,648\,680\,877\,747\,112 = E,$$

$$\Delta H = Er - {\cdot}008\,533\,556\,834\,106\,5928r^2 + {\cdot}012\,468\,474\,707\,611\,2995r^3$$
$$- {\cdot}012\,117\,650\,704\,431\,3617r^4 + {\cdot}007\,892\,402\,268\,970\,72r^5$$
$$- {\cdot}003\,082\,548\,289\,9949r^6 + {\cdot}000\,234\,604\,002\,670r^7 + {\cdot}000\,519\,783\,660\,54r^8$$
$$- {\cdot}000\,322\,835\,397\,25r^9 + {\cdot}000\,062\,555\,0066r^{10} + {\cdot}000\,025\,530\,8798r^{11}$$
$$- {\cdot}000\,019\,690\,38r^{12} + {\cdot}000\,003\,669\,787r^{13} + {\cdot}000\,001\,338\,32r^{14} - \text{etc.}$$

For $t = 2{\cdot}5$, $\epsilon^{-t^2} = 0{\cdot}001\,930\,454\,136\,227\,709\,242\,213\,515$;

$$\Delta G = -r\epsilon^{-t^2}\{1 - 2{\cdot}5r + 3{\cdot}8\dot{3}r^2 - 3{\cdot}958\dot{3}r^3 + 2{\cdot}808\dot{3}r^4 - 1{\cdot}284\,7\dot{2}r^5 + 0{\cdot}249\,00\dot{7}\,936\dot{5}r^6$$
$$+ {\cdot}119\,667\,6\dot{5}8\,730\dot{1}r^7 - {\cdot}114\,900\,242\,504\,41r^8 + {\cdot}036\,175\,870\,8113r^9$$
$$+ {\cdot}002\,358\,280\,2229r^{10} - {\cdot}006\,463\,809\,307r^{11} + {\cdot}003\,355\,4256r^{12}$$
$$+ {\cdot}000\,231\,605\,1738r^{13} - \}.$$

and $\frac{2}{\sqrt{\pi}}\epsilon^{-t^2} = 0{\cdot}002\,178\,284\,230\,352\,709\,720\,3867 = E$; $H = 0{\cdot}999\,593\,047\,982\,555\,0361$;

$$\Delta H = Er - {\cdot}005\,445\,710\,575\,881\,774\,35r^2 + {\cdot}008\,350\,089\,549\,685\,3876r^3$$
$$- {\cdot}008\,622\,375\,078\,479\,476r^4 + {\cdot}006\,117\,348\,213\,573\,86r^5 - {\cdot}002\,798\,490\,157\,0504r^6$$
$$+ {\cdot}000\,542\,410\,061\,328r^7 + {\cdot}000\,260\,670\,173\,895r^8 - {\cdot}000\,250\,285\,386\,31r^9$$
$$+ {\cdot}000\,078\,801\,328\,91r^{10} + {\cdot}000\,005\,137\,0046r^{11} - {\cdot}000\,014\,080\,014r^{12} + {\cdot}000\,007\,309\,07r^{13}$$
$$+ {\cdot}000\,000\,5044r^{14} - \text{etc.}$$

For $t = 2{\cdot}6$, $\epsilon^{-t^2} = 0{\cdot}001\,159\,229\,173\,904\,591\,150\,012$;

$$\Delta G = -r\epsilon^{-t^2}\{1 - 2{\cdot}6r + 4{\cdot}17\dot{3}r^2 - 4{\cdot}558\dot{6}r^3 + 3{\cdot}489\,01\dot{3}r^4 - 1{\cdot}808\,167\dot{1}r^5 + 0{\cdot}512\,492\,3\dot{9}3\,650\dot{7}r^6$$
$$+ {\cdot}054\,344\,32\dot{5}\,079\,3\dot{6}r^7 - {\cdot}131\,050\,242\,144\,62r^8 + {\cdot}058\,484\,912\,567\,76r^9$$
$$- {\cdot}006\,202\,829\,135\,64r^{10} - {\cdot}010\,449\,018\,028\,06r^{11} + {\cdot}005\,053\,337\,26r^{12}$$
$$+ {\cdot}002\,601\,512\,3865r^{13} - \}.$$

and $\frac{2}{\sqrt{\pi}}\epsilon^{-t^2} = 0{\cdot}001\,308\,050\,049\,723\,251\,542\,496 = E$;

$$\Delta H = Er - {\cdot}003\,400\,930\,129\,280\,4540r^2 + {\cdot}005\,458\,928\,874\,178\,370r^3 - {\cdot}005\,962\,964\,160\,005\,063r^4$$
$$+ {\cdot}004\,563\,804\,064\,151\,75r^5 - {\cdot}002\,365\,173\,079\,5968r^6 + {\cdot}000\,670\,365\,700\,9977r^7$$
$$+ {\cdot}000\,071\,085\,097\,12r^8 - {\cdot}000\,171\,420\,2756r^9 + {\cdot}000\,076\,501\,193r^{10} - {\cdot}000\,008\,113\,61r^{11}$$
$$- {\cdot}000\,013\,6678r^{12} + {\cdot}000\,006\,610r^{13} + {\cdot}000\,003\,4029r^{14} - \text{etc.}$$

For $t = 2{\cdot}8$, $\epsilon^{-t^2} = 0{\cdot}000\ 393\ 669\ 040\ 655\ 078\ 210\ 9805$;

$$\Delta G = -r\epsilon^{-t^2}\{1 - 2{\cdot}8r + 4{\cdot}89\dot{3}r^2 - 5{\cdot}917\dot{3}r^3 + 5{\cdot}159\ 41\dot{3}r^4 - 3{\cdot}237\ 496\dot{8}r^5 + 1{\cdot}361\ 565\ 7\dot{6}5\ 079\dot{3}r^6$$
$$- 0{\cdot}259\ 346\ 70\dot{2}r^7 - {\cdot}103\ 377\ 617\ 382\ 716\ 05r^8 + {\cdot}103\ 997\ 546\ 129\ 383r^9$$
$$- {\cdot}036\ 027\ 867\ 912\ 33r^{10} + {\cdot}001\ 055\ 801r^{11} + {\cdot}004\ 625\ 1919r^{12} - {\cdot}001\ 989\ 645r^{13}, \text{ etc.}\}.$$

and $\frac{2}{\sqrt{\pi}}\epsilon^{-t^2} = 0{\cdot}000\ 444\ 207\ 944\ 205\ 666\ 629\ 3623 = E$;

$$\Delta H = Er - 0{\cdot}001\ 243\ 782\ 243\ 775\ 866\ 56r^2 + {\cdot}002\ 173\ 657\ 540\ 313\ 0620r^3$$
$$- {\cdot}002\ 628\ 526\ 475\ 179\ 665r^4 + {\cdot}002\ 291\ 852\ 390\ 107\ 31r^5 - {\cdot}001\ 438\ 121\ 837\ 3856r^6$$
$$+ {\cdot}000\ 604\ 818\ 329\ 407r^7 - {\cdot}000\ 115\ 203\ 865\ 431r^8 - {\cdot}000\ 045\ 921\ 158\ 89r^9$$
$$+ {\cdot}000\ 046\ 196\ 536\ 17r^{10} - {\cdot}000\ 016\ 003\ 8651r^{11} + {\cdot}000\ 000\ 468\ 995r^{12} + {\cdot}000\ 002\ 054\ 511r^{13}$$
$$- {\cdot}000\ 000\ 088\ 38r^{14} + \text{etc.}$$

Lastly, for $t = 3$, $\epsilon^{-t^2} = 0{\cdot}000\ 123\ 409\ 804\ 086\ 679\ 549\ 4976$;

$$\Delta G = -r\epsilon^{-t^2}\{1 - 3r + 5{\cdot}\dot{6}\ r^2 - 7{\cdot}5\ r^3 + 7{\cdot}3r^4 - 5{\cdot}3r^5 + 2{\cdot}8\dot{0}4\ 761\dot{9}r^6 - 0{\cdot}967\ 857\tfrac{1}{7}r^7 + {\cdot}099\ \dot{8}67\ 72\dot{4}r^8$$
$$+ {\cdot}112\tfrac{1}{7}r^9 - {\cdot}077\ \dot{5}10\ 82\dot{2}r^{10} + {\cdot}021\ 7\dot{6}4\ 069\dot{2}r^{11} + {\cdot}000\ 8\dot{8}6\ 058\dot{3}r^{12}$$
$$- {\cdot}003\ 249\ 626\ 464\ 0122r^{13} + \text{ etc.}\}.$$

and $\frac{2}{\sqrt{\pi}}\epsilon^{-t^2} = 0{\cdot}000\ 139\ 253\ 051\ 946\ 747\ 853\ 89 = E$; $H = 0{\cdot}999\ 977\ 909\ 503\ 001\ 4145$;

$$\Delta H = Er - {\cdot}000\ 417\ 759\ 155\ 840\ 243\ 561\ 67r^2 + {\cdot}000\ 789\ 100\ 627\ 698\ 237\ 8386r^3$$
$$- {\cdot}001\ 044\ 397\ 889\ 600\ 608\ 904r^4 + {\cdot}001\ 016\ 547\ 279\ 211\ 259\ 33r^5$$
$$- {\cdot}000\ 738\ 041\ 175\ 317\ 7636r^6 + {\cdot}000\ 390\ 571\ 655\ 222\ 069r^7 - {\cdot}000\ 134\ 777\ 060\ 991\ 32r^8$$
$$+ {\cdot}000\ 013\ 906\ 885\ 4788r^9 + {\cdot}000\ 015\ 616\ 285\ 111r^{10} - {\cdot}000\ 010\ 793\ 618\ 59r^{11}$$
$$+ {\cdot}000\ 003\ 030\ 713\ 07r^{12} + {\cdot}000\ 000\ 123\ 386r^{13} - {\cdot}000\ 000\ 452\ 53r^{14} + \text{ etc.}$$

These data will enable anyone to verify the table, and also to recompute to the like degree of accuracy KRAMP's first Table of the values of G. Any value of G may also be found for verification by multiplying $1 - H$ by $\frac{1}{2}\sqrt{\pi}$.

The constant ρ and its derivatives.

20. The value of $t = \rho$, in the solution of the equation—

$$t - \frac{t^3}{3} + \frac{1}{1.2}\frac{t^5}{5} + \frac{1}{1.2.3}\frac{t^7}{7} - \text{etc.} = \frac{\sqrt{\pi}}{4}; \text{ or } \frac{2}{\sqrt{\pi}}\left(t - \frac{t^3}{3} + \frac{t^5}{2.5} - \frac{t^7}{3!7} + \right) = \frac{1}{2} \qquad (37)$$

is of importance, as it enters into the coefficients of various formulæ. BESSEL employed the value 0·476 9364, ENCKE, followed by DE MORGAN, uses 0·476 9360, and AIRY gives 0·476 948.* To obtain this value with extreme accuracy, we may proceed thus: Since $0{\cdot}475 = \frac{1}{2}(1 - \frac{1}{20})$ and $0{\cdot}475^2 = \frac{1}{5} + \frac{1}{5.8} + \frac{1}{40.40}$, the computation of the series for this value is comparatively easy, and gives—for $t = {\cdot}475$—

* It seems strange that the late Astronomer Royal, so late as 1861, should have adopted a value differing from that so generally recognised as correct at least to six decimal figures; he gives its reciprocal also as 2·096 665 (*Theory of Errors*, pp. 23, 24). LAPLACE (*Théorie Anal. des Probabilités*, 2e ed., p. 238), in one of the very few examples he gives, makes $t^2 = {\cdot}210\ 2497$, which would give $\rho = {\cdot}45853$. M. POISSON, also (*Connaissance des Temps*, 1832, Add. p. 20), gives ·47414 for the value of ρ, and ·67236 for that of $\rho\sqrt{\pi}$, and again (*Rech. sur la Prob. des Jugements*, p. 208), he has ·4765 and ·6739 for the same quantities. GAUSS (*Werke*, Bd. iv, S. 110) gave the value as ·476 9363, which is correct to the nearest figure in the seventh place. Lastly, O. BYRNE (*Dual Arithmetic*, p. 200) finds 0·476 936 2744, which errs only in the last two decimal figures.

$$\frac{2}{\sqrt{\pi}}\int_{t}^{\infty}\epsilon^{-t^2}dt = 0{\cdot}498\ 258\ 053\ 711\ 787\ 564\ 127\ 43 = \text{H}.$$

For the same value of t, we have—

$$\frac{2}{\sqrt{\pi}}\epsilon^{-t^2} = 0{\cdot}900\ 466\ 098\ 615\ 398\ 685\ 314\ 176 = \epsilon\,,$$

and KRAMP's formula (36) for a difference r, becomes—

$$\begin{aligned}\tfrac{1}{2}-\text{H} &= \epsilon r(1-0{\cdot}475r-0{\cdot}182\ 91\dot{6}r^2+0{\cdot}201\ 776\ 041\dot{6}r^3+0{\cdot}016\ 537\ 552r^4\\ &\qquad -0{\cdot}056\ 42539r^5+0{\cdot}00372r^6+\text{etc.})\\ &= 0{\cdot}900\ 466\ 098\ 615\ 398\ 685\ 314\ 176r-0{\cdot}427\ 721\ 396\ 842\ 314\ 8755r^2\\ &\qquad -0{\cdot}164\ 710\ 257\ 205\ 0667r^3+0{\cdot}181\ 692\ 485\ 0336r^4+0{\cdot}014\ 891\ 505r^5\\ &\qquad -0{\cdot}050\ 809r^6+0{\cdot}00335r^7+\text{etc.}\end{aligned}$$

Using the first three terms, and taking $r = {\cdot}001\ 936$ as a first approximation, we obtain $\text{H}'-\text{H} = +{\cdot}001\ 741\ 699\ 03$, etc. But $\frac{1}{2}-\text{H} = {\cdot}001\ 741\ 946\ 29$, and the difference of these is $\frac{1}{2}-\text{H}' = +{\cdot}000\ 000\ 247\ 26$. The value of $\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$ for $t = 0{\cdot}476\ 936$ is $0{\cdot}898\ 80814$. Hence the correction is $\frac{{\cdot}002\ 4726}{8988} = +0{\cdot}000\ 000\ 275$, and the new value of ρ is $0{\cdot}476\ 936 + {\cdot}000\ 000\ 275 = {\cdot}476\ 936\ 275$; and from this,—taking in the higher powers of r,—we readily arrive at the value, correct to the twenty-fourth place of decimals, viz.:—

$$\rho = 0{\cdot}476\ 936\ 276\ 204\ 469\ 873\ 383\ 506.$$

Otherwise, we may form a difference-formula for the computation of this and other values of t corresponding to definite values of H. Thus let H be the tabular or computed value corresponding to t, and H′ the value for which $t' = t+\Delta t$ is sought. Put $h = \frac{1}{2}\sqrt{\pi}(\text{H}'-\text{H})\epsilon^{-t^2}$. Then—

$$\Delta t = h\left(1+ht+\frac{4t^2+1}{3}h^2+\frac{12t^3+7t}{6}h^3+\frac{96t^4+92t^2+7}{2.3.5}h^4+\frac{480t^5+652t^3+127t}{3.5.6}h^5+\text{etc.}\right). \qquad (38)$$

Using the above value of H for $t = {\cdot}475$, we find $h = {\cdot}001\ 934\ 494\ 025\ 806\ 1229$, and this series becomes—

$$\Delta t = h(1+{\cdot}475h+{\cdot}634\ 1\dot{6}h^2+{\cdot}768\ 510\ 41\dot{6}h^3+1{\cdot}088\ 151\ 25h^4+1{\cdot}575\ 641\ 961\ 80\dot{5}h^5+\ldots),$$

and this gives at once the value of ρ correctly to seventeen figures. When h is very small, the first three terms of (38) will usually be sufficient to determine the values of t corresponding to $\text{H} = 0{\cdot}1$, $0{\cdot}2$, $0{\cdot}3$, etc., as given below, § 23.

21. The following table contains the values of the factors dependent on this constant, ρ, together with some others used in Probabilities,* with their logarithms, computed to a degree of accuracy far beyond what can be required.

* These constants will be met with, among other places, in BESSEL's *Fundamenta Astron.*, p. 18; and *Ueber d. Bahn des Olberschen Kometen*, in *Abh. d. Math. Kl. d. Königl. Preuss. Akad.*, 1812–13, S. 142; DE MORGAN's *Theory of Probab.*, §§ 68, 100, 116, 150, 152, etc.; ENCKE, in *Berl. Ast. Jahrb.*, 1834, Ss. 270, 293, 298; GAUSS, *Werke*, Bd. iv, S. 6; AIRY, *Theory of Errors*, pp. 23, 24; POISSON, *Rech. sur la Probab. des Jugements*, p. 176, etc.

TABLE.

	Constants.	Values of Constants.	Logarithms.
1	ρ	0·476 936 276 204 469 873 383 51	$\bar{1}$·678 460 356 521 217 913 230 78
2	$\frac{1}{\rho}$	2·096 716 165 015 061 071 615 78	0·321 539 643 478 782 086 769 22
3	ρ^2	0·227 468 211 559 786 375 973 25	$\bar{1}$·356 920 713 042 435 826 461 56
4	$\rho\sqrt{2}$	0·674 489 750 196 035 151 103 81	$\bar{1}$·828 975 354 353 208 510 837 65
5	$2\rho\sqrt{\pi}$	1·690 695 078 790 009 806 981 30	0·228 065 288 532 266 035 620 15
6	$\frac{2}{\sqrt{\pi}}\rho$	0·538 164 958 101 235 048 729 82	$\bar{1}$·730 915 415 838 132 181 268 88
7	$\frac{2}{\sqrt{\pi}}\rho^2$	0·256 670 391 159 638 137 627 19	$\bar{1}$·409 375 772 459 350 094 499 66
8	$\frac{1}{\rho\sqrt{\pi}}$	1·182 945 419 957 695 955 821 42	0·072 964 707 131 715 159 593 59
9	$\frac{1}{2\rho\sqrt{\pi}}$	0·591 472 709 978 847 977 910 71	$\bar{1}$·771 934 711 467 733 964 379 85
10	$\rho\sqrt{\pi}$	0·845 347 589 395 004 903 490 65	$\bar{1}$·927 035 292 868 284 840 406 41
11	$\frac{1}{\rho\sqrt{2}}$	1·482 602 218 505 601 860 540 58	0·171 024 645 646 791 489 162 35
12	$\rho\sqrt[6]{\pi}$	0·577 189 827 811 086 284 473 01	$\bar{1}$·761 318 668 636 906 888 955 99
13	$\rho\sqrt[10]{\frac{\pi}{4}}$	0·465 553 230 574 244 418 753 06	$\bar{1}$·667 969 344 657 835 059 623 16
14	$\rho\sqrt{\pi-2}$	0·509 584 182 684 138 078 029 73	$\bar{1}$·707 215 939 186 776 502 110 25
15	$\rho\sqrt{\frac{15\pi-8}{36}}$	0·497 198 854 778 314 121 494 65	$\bar{1}$·696 530 119 639 696 588 914 00
16	$\rho\sqrt{\frac{945\pi-128}{1600}}$	0·635 508 087 011 832 529 750 44	$\bar{1}$·803 121 081 439 621 574 379 57
17	$\rho\sqrt{\frac{4}{3}}$	0·550 718 574 905 896 772 795 56	$\bar{1}$·740 929 724 825 367 889 797 00
18	$\rho\sqrt[4]{\frac{4}{3}}$	0·512 501 381 805 211 150 143 34	$\bar{1}$·709 695 040 673 292 901 513 89
19	$\rho\sqrt{\frac{113}{45}}$	0·755 776 391 184 821 580 506 05	$\bar{1}$·878 393 321 375 255 934 940 23
20	$\rho\sqrt[6]{\frac{8}{15}}$	0·429 497 009 734 013 564 961 27	$\bar{1}$·632 960 144 510 594 970 490 77
21	ϵ^{ρ^2}	1·255 417 531 354 680 356 016 89	0·098 788 189 088 816 702 448 09
22	$\epsilon^{\rho^2}\sqrt{\pi}$	2·225 169 637 943 592 189 588 00	0·347 363 125 435 823 629 623 72
23	$\epsilon^{-\rho^2}$	0·796 547 742 105 315 688 192 06	$\bar{1}$·901 211 810 911 183 297 551 91
24	$\frac{2}{\sqrt{\pi}}\epsilon^{-\rho^2}$	0·898 807 877 788 607 267 593 84	$\bar{1}$·953 666 870 228 097 565 590 02
25	$\frac{\epsilon}{2\sqrt{2}}$	0·961 057 757 039 779 206 215 42	$\bar{1}$·982 749 488 407 280 034 830 52
26	$\frac{\epsilon}{\sqrt{2\pi}}$	1·084 437 551 419 227 546 611 58	0·035 204 547 724 194 302 868 63
27	$\sqrt{\frac{\pi}{2}}$	1·253 314 137 315 500 215 207 88	0·098 059 938 515 076 329 568 76
28	$\sqrt{\frac{2}{\pi}}$	0·797 884 560 802 865 355 879 89	$\bar{1}$·901 940 061 484 923 670 431 24
29	$\frac{1}{\sqrt{\pi}}$	0·564 189 583 547 756 286 948 08	$\bar{1}$·751 425 063 652 933 072 824 37
30	$\frac{2}{\sqrt{\pi}}$	1·128 379 167 095 512 573 896 16	0·052 455 059 316 914 268 038 10
31	$\frac{1}{2}\sqrt{\pi}$	0·886 226 925 452 758 013 649 08	$\bar{1}$·947 544 940 683 085 731 961 90

22. In the theory of Errors of Observations, we may state the proportions of the different constants for 'modulus,' 'mean error,' 'error of mean square,' and 'probable error,' as in the adjoining table.* And the ordinary relations of 'mean' or average error, A (double the mean risk); weight of an observation, or square of the number of observations divided by twice the sum of the squares of the errors, W; modulus, M; the error of mean square, S; and probable error, E,—are expressed by the equations,—

	Modulus.	Mean Error.	Error of Mean Square.	Probable Error.
In terms of modulus	1	$\frac{1}{\sqrt{\pi}}$	$\frac{1}{\sqrt{2}}$	ρ
In terms of mean error	$\sqrt{\pi}$	1	$\sqrt{\frac{\pi}{2}}$	$\rho\sqrt{\pi}$
In terms of error of mean square	$\sqrt{2}$	$\sqrt{\frac{2}{\pi}}$	1	$\rho\sqrt{2}$
In terms of probable error	$\frac{1}{\rho}$	$\frac{1}{\rho\sqrt{\pi}}$	$\frac{1}{\rho\sqrt{2}}$	1

$$\left.\begin{array}{ll} M=A\sqrt{\pi}=S\sqrt{2}=\frac{E}{\rho}=\frac{1}{\sqrt{W}}. & A=\frac{M}{\sqrt{\pi}}=S\sqrt{\frac{2}{\pi}}=\frac{E}{\rho\sqrt{\pi}}=\frac{1}{\sqrt{\pi W}}. \\ E=M\rho=A\rho\sqrt{\pi}=S\rho\sqrt{2}=\frac{\rho}{W}. & S=\frac{M}{\sqrt{2}}=A\sqrt{\frac{\pi}{2}}=\frac{E}{\rho\sqrt{2}}=\frac{1}{\sqrt{2W}}, \\ \multicolumn{2}{c}{\text{and } W=\frac{1}{M^2}=\frac{1}{\pi A^2}=\frac{1}{2S^2}=\frac{\rho^2}{E^2}.} \end{array}\right\} \quad (39)$$

The values of the constants are found in the table above; but for approximations, that are occasionally useful, the following may be given :—

For $\sqrt{\pi}$ we may use $\frac{296}{167}=1{\cdot}772\,455$, or roughly $\frac{709}{400}$;

for $\sqrt{\frac{2}{\pi}}$ „ „ $\frac{679}{851}=0{\cdot}797\,8848$, or $\frac{379}{475}=\cdot797\,895$, or $\frac{75}{94}=\cdot797\,87$, or $\frac{399}{500}$;

for $\sqrt{2}$ „ „ $\frac{577}{408}=1{\cdot}414\,2156$, or $\frac{239}{169}=1{\cdot}414\,201$, or $\frac{99}{70}=1{\cdot}414\,286$;

for ρ „ „ $\frac{455}{954}=0{\cdot}476\,939$, or $\frac{300}{629}=\cdot476\,9475$, or $\frac{31}{65}=\cdot476\,923$;

for $\rho\sqrt{2}$ „ „ $\frac{661}{980}=0{\cdot}674\,4898$, or $\frac{201}{298}=\cdot674\,497$, or $\frac{29}{43}=\cdot674\,45$;

for $\rho\sqrt{\pi}$ „ „ $\frac{645}{763}=0{\cdot}845\,3472$, or $\frac{82}{97}=\cdot845\,36$, or $\frac{71}{84}=\cdot845\,24$;

and for ρ^2 „ „ $\frac{53}{233}=0{\cdot}227\,468$, or $\frac{48}{211}=\cdot227\,488$, or $\frac{5}{22}=\cdot227\,27$;

Whence,— $M=\frac{296}{167}A=\frac{239}{169}S$, or $\frac{99}{70}S=\frac{629}{300}E$, or $\frac{65}{31}E=\frac{1}{\sqrt{W}}$.

* Conf. AIRY's *Theory of Errors*, p. 24; Galloway's *Treat. on Probability*, §§ 145–148, pp. 194–197; De Morgan's *Essay*. p. 139.

$$A = \frac{167}{296}M = \frac{679}{851}S, \text{ or } \frac{75}{94}S = \frac{763}{645}E, \text{ or } \frac{97}{82}E = \frac{167}{296\sqrt{W}}.$$

$$E = \frac{455}{954}M, \text{ or } \frac{31}{65}M = \frac{645}{763}A, \text{ or } \frac{82}{97}A = \frac{661}{980}S = \frac{300}{629\sqrt{W}} \text{ or } \frac{31}{65\sqrt{W}}.$$

$$S = \frac{408}{577}M, \text{ or } \frac{70}{99}M = \frac{851}{679}A, \text{ or } \frac{94}{75}A = \frac{298}{201}E = \frac{169}{239\sqrt{W}}.$$

$$\text{and } W = \frac{1}{M^2} = \frac{113}{355A^2} \text{ or } \frac{7}{22A^2} = \frac{1}{2S^2} = \frac{53}{233E^2} \text{ or } \frac{5}{22E^2}.$$

23. Besides ρ, other values of t corresponding to certain definite values of H may occasionally be required,* and the extent of the table now given will enable us to determine them with a high degree of accuracy by simple interpolation †; thus :—

For H = 0·1,	t = 0·088 885 991	.. log $\bar{2}$·948 832 9230	... 0·186 367 523.ρ
0·2,	0·179 143 455	$\bar{1}$·253 200 9459	0·375 512 978.ρ
0·3,	0·272 462 716	$\bar{1}$·435 303 8936	0·571 272 788.ρ
0·4,	0·370 807 149	$\bar{1}$·569 148 0986	0·777 377 028.ρ
0·5,	0·476 936 276	$\bar{1}$·678 460 3565	1·000 000 000.ρ
0·6,	0·595 116 079	$\bar{1}$·744 601 6843	1·247 789 503.ρ
0·7,	0·732 869 079	$\bar{1}$·865 026 3985	1·536 618 445.ρ
0·8,	0·906 193 802	$\bar{1}$·957 221 0875	1·900 031 193.ρ
0·9,	1·163 087 153	0·065 612 2587	2·438 663 635.ρ
1·0,	∞	∞	∞

Construction of the Table.

24. In both divisions of the general integral the factor ϵ^{-t^2} forms a multiplier. Assistance in obtaining the values of this factor might have been derived from the extensive tables of ϵ^{-x} by Prof. F. W. NEWMAN and Mr GLAISHER,‡ had they been in existence when the following table was begun. But the interpolation for values of ϵ^{-t^2}, by means of the formula—

$$\epsilon^{-x\pm h} = \epsilon^{-x}\left\{1 \pm \frac{h}{1} + \frac{h^2}{1.2} \pm \frac{h^3}{1.2.3} + \text{etc.}\right\} \tag{40}$$

is somewhat laborious, since h in this case has the form of $2xh + h^2$. As the factor in the function H is the multiple $\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$, it is occasionally convenient to find its value logarithmically, and also as part of the computation of the value of the function, the former proving a check on the working for the latter.§ In the first part of the

* GAUSS, *Bestimm. d. Genauigkeit d. Beobacht.*, § 2 ; *Werke*, Bd. iv, S. 110.

† Or, the difference formula (38), given above, § 20, may be used to find these values.

‡ *Trans. Camb. Phil. Soc.*, vol. xiii, (1883), pp. 145–272.

§ If we compute in succession, as is naturally the easiest method, the terms of the expression—

$$\frac{2}{\sqrt{\pi}}\left(1 + t - t^2 - t^3 + \frac{t^4}{2} + \frac{t^5}{2} - \frac{t^6}{3!} - \frac{t^7}{3!} + \frac{t^8}{4!} + \frac{t^9}{4!} - \text{etc.}\right)$$

the sum of the 1st, 3rd, 5th, 7th, etc., terms will give the value of $\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$, whilst the sum of the quotients of the 2nd, 4th, 6th, 8th, etc., terms, divided respectively by 1, 3, 5, 7, etc., will give the value of H.

tables the values of this factor are given for every value of t, and, at larger intervals, from $t=1\cdot25$ to $t=6\cdot0$ (on p. 295). The following values were also computed with extreme accuracy :—

x.	ϵ^{-x}.	$\frac{2}{\sqrt{\pi}}\epsilon^{-x}$.
0	1·000	1·128 379 167 095 512 573 896 158 903
$\frac{1}{2}$	·606 530 659 712 633 423 603 799 534 990	0·684 396 560 624 433 066 358 502 37
1	·367 879 441 171 442 321 595 523 770 161	0·415 107 497 420 594 703 340 268 249
2	·135 335 283 236 612 691 893 999 494 972	0·152 709 514 177 164 314 421 873 367
3	·049 787 068 367 863 942 979 342 415 650	0·056 178 690 737 057 656 594 924 613
4	·018 315 638 888 734 180 293 718 021 273	0·020 666 985 354 092 053 857 068 941
5	·006 737 946 999 085 467 096 636 048 423	0·007 602 959 022 761 767 784 966 646
6	·002 478 752 176 666 358 423 045 167 431	0·002 796 972 316 542 974 354 763 250
7	·000 911 881 965 554 516 208 003 136 084	0·001 028 948 612 781 823 885 494 178
8	·000 335 462 627 902 511 838 821 389 126	0·000 378 529 040 664 308 164 933 856
9	·000 123 409 804 086 679 549 497 636 691	0·000 139 253 051 946 747 853 890 418
10	·000 045 399 929 762 484 851 535 591 516	0·000 051 228 334 931 587 428 772 169

25. The first part of the table contains the values of H from $t=0$ to $t=1\cdot250$, at intervals of ·001 to nine places of decimals, together with the first and second differences, and the corresponding values of $\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$.* These values were computed in 1862, by using the general series for intervals of ·02, and interpolating for the intermediate values with six or more orders of differences. The second part contains the values from $t=1\cdot00$ to $t=3\cdot00$, computed recently, to fifteen decimal figures,—(1) from $t=1\cdot000$ to $t=1\cdot500$ at intervals of ·001, and (2) from $t=1\cdot5$ to $t=3\cdot0$ at intervals of ·002, with four orders of differences. In the last column of this portion of the table are given the corresponding values of log $\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}+10$, to sixteen decimal places. And (3) lastly, values of H and G from 3·0 to 6·0 are appended,† computed by means of LAPLACE's fraction (§ 9),—the values of L (16) being preserved. These would enable us to extend the general table still farther if required.

* The differences of these values have been omitted from want of room on the page. The differences given throughout the tables are stated to the nearest figure in the last place, being taken from the computations.

† Mr J. W. L. GLAISHER's table (referred to above, § 4) of the values of G from $t=3\cdot00$ to $4\cdot50$ (*Phil. Mag.*, 4th ser., 1871, vol. xlii. p. 436) is computed for differences of 0·01 and to seven significant figures, that is from eleven to fourteen decimal places; the appended table gives the values computed to fifteen places. But the values of L would enable us to carry them to a much larger number of figures.

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (1) FROM $t=0$ TO $t=1·250$.

$t=0·0$] [to ·099.

t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$	t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$
0·000	·000 000 000	1 128 379	00	1·128 379 167	0·050	·056 371 978	1 125 505	113	1·125 561 742
1	001 128 379	377	02	378 039	51	057 497 483	390	15	448 066
2	002 256 755	372	05	374 654	52	058 622 873	273	17	332 151
3	003 385 127	365	07	369 012	53	059 748 146	154	19	213 998
4	004 513 493	1 128 356	09	361 113	54	060 873 300	1 125 032	22	093 606
0·005	·005 641 849	345	11	1·128 350 958	0·055	·061 998 ~~609~~	1 124 909	124	1·124 970 978 1+332
6	006 770 194	331	14	338 546	56	063 123 242	783	26	846 113
7	007 898 525	316	16	323 878	57	064 248 024	655	28	719 012
8	009 026 841	298	18	306 953	58	065 372 679	524	30	589 677
9	010 155 138	1 128 277	20	287 772	59	066 497 203	1 124 391	33	458 108
0·010	·011 283 416	255	23	1·128 266 335	0·060	·067 621 594	256	135	1·124 324 305
11	012 411 670	230	25	242 641	61	068 745 851	119	37	188 270
12	013 539 900	203	27	216 692	62	069 869 970	1 123 980	39	050 004
13	014 668 103	173	29	188 487	63	070 993 950	838	42	1·123 909 506
14	015 796 276	1 128 142	32	158 026	64	072 117 788	1 123 694	44	766 779
Γ6 0·015	·01[illegible] 924 418	108	34	1·128 125 310	0·065	·073 241 483	548	146	1·123 621 822
16	018 052 526	072	36	090 339	66	074 365 031	400	48	474 637
17	019 180 598	034	38	053 126	67	075 488 431	250	51	325 225
18	020 308 632	1 127 993	41	013 631	68	076 611 681	097	53	173 586
19	021 436 625	1 127 950	43	1·127 971 896	69	077 734 778	1 122 942	55	019 722
0·020	·022 564 575	905	45	1·127 927 906	0·070	·078 857 720	785	157	1·122 863 633
21	023 692 480	858	47	881 662	71	079 980 504	625	59	705 321
22	024 820 337	808	50	833 164	72	081 103 130	464	62	544 785
23	025 948 145	756	52	782 412	73	082 225 593	300	64	382 028
24	027 075 901	1 127 702	54	729 408	74	083 347 893	1 122 134	66	217 050
0·025	·028 203 603	646	56	1·127 674 150	0·075	·084 470 027	1 121 965	168	1·122 049 852
26	029 331 249	587	59	616 641	76	085 591 992	795	71	1·121 880 435
27	030 458 836	526	61	556 878	77	086 713 787	622	73	708 801
28	031 586 362	463	63	494 865	78	087 835 409	447	75	534 949
29	032 713 825	1 127 398	65	430 599	79	088 956 856	1 121 270	77	358 882
0·030	·033 841 222	330	68	1·127 364 083	0·080	·090 078 126	091	179	1·121 180 600
31	034 968 552	260	70	295 316	81	091 199 216	1 120 909	82	000 105
32	036 095 812	188	72	224 298	82	092 320 125	725	84	1·120 817 397
33	037 223 000	113	74	151 031	83	093 440 850	539	86	632 477
34	038 350 114	1 127 037	77	075 514	84	094 561 390	1 120 351	88	445 347
0·035	·039 477 150	1 126 958	79	1·126 997 749	0·085	·095 681 740	160	190	1·120 256 008
36	040 604 108	877	81	917 735	86	096 801 901	1 119 968	93	064 460
37	041 730 985	793	83	835 473	87	097 921 869	773	95	1·119 870 706
38	042 857 778	708	86	750 963	88	099 041 641	576	97	674 746
39	043 984 486	1 126 620	88	664 207	89	100 161 217	1 119 377	99	476 581
0·040	·045 111 106	530	90	1·126 575 204	0·090	·101 280 594	175	201	1·119 276 213
41	046 237 636	437	92	483 955	91	102 399 769	1 118 971	04	073 642
42	047 364 073	343	95	390 461	92	103 518 740	766	06	1·118 868 870
43	048 490 416	246	97	294 722	93	104 637 506	557	08	661 899
44	049 616 662	1 126 147	99	196 738	94	105 756 064	1 118 347	10	452 728
0·045	·050 742 809	045	101	1·126 096 511	0·095	·106 874 411	135	212	1·118 241 361
46	051 868 854	1 125 942	104	1·125 994 041	96	107 992 546	1 117 920	15	027 797
47	052 994 796	836	106	889 329	97	109 110 466	703	17	1·117 812 039
48	054 120 632	728	108	782 374	98	110 228 169	484	19	594 086
49	055 246 360	1 125 618	110	673 179	99	111 345 653	1 117 263	21	373 942

Table of the Values of $H = \frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2} dt$. **(1) From** $t = 0$ **to** $t = 1·250$.

$t =$ ·100] ['199

t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$	t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$
0·100	·112 462 916	1 117 040	223	1·117 151 607	0·150	·167 995 971	1 103 108	331	1·103 274 127
1	113 579 956	6 814	26	6 927 082	51	169 099 080	2 775	33	2 942 091
2	114 696 769	6 586	28	6 700 369	52	170 201 855	2 440	35	2 607 950
3	115 813 356	6 356	30	6 471 469	53	171 304 295	2 103	37	2 271 706
4	116 929 712	1 116 124	32	6 240 383	54	172 406 398	1 101 763	39	1 933 361
0·105	·118 045 836	5 890	234	1·116 007 113	0·155	·173 508 161	1 422	341	1·081 592 916
6	119 161 725	5 653	37	5 771 660	56	174 609 583	1 078	44	1 250 374
7	120 277 378	5 414	39	5 534 026	57	175 710 661	0 733	46	1·080 905 736
8	121 392 792	5 173	41	5 294 212	58	176 811 394	0 385	48	0 559 006
9	122 507 966	1 114 930	43	5 052 220	59	177 911 778	1 100 035	50	0 210 184
0·110	·123 622 896	4 685	245	1·114 808 050	0·160	·179 011 813	1 099 683	352	1·099 850 273
11	124 737 581	4 438	47	4 561 705	61	180 111 496	9 329	54	9 506 274
12	125 852 019	4 188	50	4 313 185	62	181 210 825	8 973	56	9 151 191
13	126 966 207	3 936	52	4 062 493	63	182 309 798	8 615	58	8 794 025
14	128 080 143	1 113 682	54	3 809 629	64	183 408 412	1 098 254	60	8 434 778
0·115	·129 193 825	3 426	256	1·113 554 596	0·165	·184 506 667	7 892	362	1·098 073 453
16	130 307 252	3 168	58	3 297 395	66	185 604 559	7 527	64	7 710 051
17	131 420 420	2 907	60	3 038 027	67	186 702 086	7 161	67	7 344 574
18	132 533 327	2 645	63	2 776 493	68	187 799 247	6 792	69	6 977 025
19	133 645 972	1 112 380	65	2 512 797	69	188 896 039	1 096 422	71	6 607 406
0·120	·134 758 352	2 113	267	1·112 246 938	0·170	·189 992 461	6 049	373	1·096 235 719
21	135 870 465	1 844	69	1 978 919	71	191 088 510	5 674	75	5 861 967
22	136 982 309	1 573	71	1 708 741	72	192 184 184	5 297	77	5 486 150
23	138 093 882	1 299	73	1 436 405	73	193 279 482	4 918	79	5 108 273
24	139 205 181	1 111 024	76	1 161 914	74	194 374 400	1 094 538	81	4 728 336
0·125	·140 316 205	0 746	278	1·110 885 270	0·175	·195 468 938	4 154	383	1·094 346 343
26	141 426 951	0 466	80	0 606 472	76	196 563 092	3 769	85	3 962 294
27	142 537 417	0 184	82	0 325 524	77	197 656 862	3 382	87	3 576 194
28	143 647 601	1 109 900	84	0 042 428	78	198 750 244	2 993	89	3 188 043
29	144 757 501	1 109 614	86	1·109 757 183	79	199 843 237	1 092 602	91	2 797 845
0·130	·145 867 115	9 325	288	1·109 469 793	0·180	·200 935 839	2 209	393	1·092 405 601
31	146 976 440	9 035	91	9 180 260	81	202 028 048	1 813	95	2 011 314
32	148 085 475	8 742	93	8 888 584	82	203 119 861	1 416	97	1 614 985
33	149 194 216	8 447	95	8 594 767	83	204 211 277	1 017	99	1 216 619
34	150 302 663	1 108 150	97	8 298 812	84	205 302 294	1 090 615	401	0 816 216
0·135	·151 410 813	1 107 851	299	1·108 000 719	0·185	·206 392 909	0212	403	1·090 413 779
36	152 518 664	7 549	301	7 700 492	86	207 483 120	1 089 806	05	0 009 310
37	153 626 214	7 246	03	7 398 131	87	208 572 927	9 399	08	1·089 602 812
38	154 733 460	6 941	06	7 093 638	88	209 662 325	8 989	10	9 194 388
39	155 840 400	1 106 633	08	6 787 016	89	210 751 315	1 088 578	12	8 783 739
0·140	·156 947 033	6 323	310	1·106 478 265	0·190	·211 839 892	8 164	414	1·088 371 168
41	158 053 356	6 011	12	6 167 389	91	212 928 056	7 748	16	7 956 578
42	159 159 367	5 697	14	5 854 388	92	214 015 805	7 331	18	7 539 970
43	160 265 064	5 381	16	5 539 264	93	215 103 135	6 911	20	7 121 348
44	161 370 445	1 105 062	18	5 222 020	94	216 190 047	1 086 490	22	6 700 713
0·145	·162 475 507	4 742	320	1·104 902 657	0·195	·217 276 536	6 066	424	1·086 278 069
46	163 580 250	4 420	23	4 581 177	96	218 362 602	5 640	26	5 853 417
47	164 684 669	4 095	25	4 257 582	97	219 448 242	5 213	28	5 426 761
48	165 788 764	3 768	27	3 931 874	98	220 533 455	4 783	30	4 998 102
49	166 892 532	1 103 439	29	3 604 055	99	221 618 238	1 084 351	32	4 567 443

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (1) FROM $t = 0$ TO $t = 1·250$.

$t = ·200$] ['299

t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$	t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$
0·200	·222 702 589	1 083 918	434	1·084 134 787	0·250	·276 326 390	1 059 749	530	1·060 014 129
1	223 786 507	3 482	36	3 700 136	51	277 386 139	9 217	32	1·059 483 195
2	224 869 989	3 044	38	3 263 493	52	278 445 356	8 683	34	8 950 409
3	225 953 033	2 605	40	2 824 860	53	279 504 039	8 148	36	8 415 774
4	227 035 638	1 082 163	42	2 384 240	54	280 562 187	1 057 610	37	7 879 294
0·205	·228 117 801	1 720	444	1·081 941 635	0·255	·281 619 797	7 071	539	1·057 340 970
6	229 199 520	1 274	46	1 497 049	56	282 676 868	6 530	41	6 800 807
7	230 280 794	0 826	48	1 050 483	57	283 733 398	5 987	43	6 258 807
8	231 361 621	0 377	50	0 601 940	58	284 789 385	5 442	45	5 714 974
9	232 441 998	1 079 925	52	0 151 423	59	285 844 827	1 054 896	47	5 169 310
0·210	·233 521 923	9 472	453	1·079 698 934	0·260	·286 899 723	4 347	548	1·054 621 819
11	234 601 395	9 016	55	9 244 477	61	287 954 071	3 797	50	4 072 505
12	235 680 411	8 559	57	8 788 053	62	289 007 868	3 245	52	3 521 369
13	236 758 970	8 100	59	8 329 665	63	290 061 113	2 691	54	2 968 415
14	237 837 070	1 077 638	61	7 869 317	64	291 113 804	1 052 136	56	2 413 647
0·215	·238 914 708	7 175	463	1·077 407 010	0·265	·292 165 939	1 578	557	1·051 857 067
16	239 991 883	6 710	65	6 942 748	66	293 217 517	1 019	59	1 298 680
17	241 068 593	6 243	67	6 476 532	67	294 268 536	0 458	61	0 738 487
18	242 144 836	5 773	69	6 008 367	68	295 318 994	1 049 895	63	0 176 492
19	243 220 609	1 075 302	71	5 538 254	69	296 368 888	1 049 330	65	1·049 612 699
0·220	·244 295 912	4 829	473	1·075 066 196	0·270	·297 418 219	8 764	566	1·049 047 110
21	245 370 741	4 354	75	4 592 197	71	298 466 982	8 195	68	8 479 729
22	246 445 095	3 877	77	4 116 258	72	299 515 177	7 625	70	7 910 559
23	247 518 973	3 399	79	3 638 382	73	300 562 803	7 053	72	7 339 603
24	248 592 371	1 072 918	81	3 158 573	74	301 609 856	1 046 480	74	6 766 865
0·225	·249 665 289	2 435	483	1·072 676 833	0·275	·302 656 336	5 904	575	1·046 192 348
26	250 737 724	1 951	85	2 193 165	76	303 702 240	5 327	77	5 616 055
27	251 809 675	1 464	87	1 707 571	77	304 747 567	4 748	79	5 037 989
28	252 881 139	0 975	88	1 220 055	78	305 792 316	4 168	81	4 458 154
29	253 952 114	1 070 485	90	0 730 620	79	306 836 483	1 043 585	82	3 876 552
0·230	·255 022 600	1 069 993	492	1·070 239 267	0·280	·307 880 068	3 001	584	1·043 293 188
31	256 092 592	9 499	94	1 069 746 001	81	308 923 069	2 415	86	2 708 065
32	257 162 091	9 002	96	9 250 823	82	309 965 484	1 827	88	2 121 186
33	258 231 093	8 504	98	8 753 737	83	311 007 311	1 238	90	1 532 554
34	259 299 598	1 068 004	500	8 254 745	84	312 048 548	1 040 646	91	0 942 172
0·235	·260 367 602	7 503	502	1·067 753 851	0·285	·313 089 194	0 053	593	1·040 350 044
36	261 435 105	6 999	04	7 251 058	86	314 129 248	1 039 459	95	1·039 756 174
37	262 502 104	6 493	06	6 746 367	87	315 168 706	8 862	96	9 160 564
38	263 568 597	5 986	08	6 239 783	88	316 207 568	8 264	98	8 563 219
39	264 634 583	1 065 476	09	5 731 308	89	317 245 832	1 037 664	600	7 964 141
0·240	·265 700 059	4 965	511	1·065 220 945	0·290	·318 283 496	7 062	602	1·037 363 333
41	266 765 024	4 452	13	4 708 697	91	319 [illegible]20 558	6 459	03	6 760 800
42	267 829 476	3 937	15	4 194 567	92	320 357 017	5 854	05	6 156 545
43	268 893 412	3 420	17	3 678 557	93	321 392 871	5 247	07	5 550 571
44	269 956 832	1 062 901	19	3 160 672	94	322 428 117	1 034 638	09	4 942 881
0·245	·271 019 733	2 380	521	1·062 640 914	0·295	·323 462 756	4 028	610	1·034 333 479
46	272 082 113	1 858	23	2 119 285	96	324 496 784	3 416	12	3 722 369
47	273 143 971	1 333	24	1 595 789	97	325 530 200	2 802	14	3 109 553
48	274 205 304	0 807	26	1 070 429	98	326 563 002	2 187	15	2 495 036
49	275 266 111	1 060 279	28	0 543 208	99	327 595 189	1 031 570	17	1 878 820

L 3

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2}dt$. (1) FROM $t=0$ TO $t=1\cdot250$.

$t = \cdot300$] [$\cdot399$

t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$	t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$
0·300	·328 626 759	1 030 951	619	1·031 260 910	0·350	·379 382 054	997 934	699	0·998 283 712
1	329 657 711	0 331	20	0 641 308	51	380 379 988	7 234	700	7 584 160
2	330 688 042	1 029 709	22	0 020 019	52	381 377 221	6 532	02	6 883 105
3	331 717 750	9 085	24	1·029 397 045	53	382 373 753	5 829	03	6 180 550
4	332 746 835	1 028 459	25	8 772 391	54	383 369 582	995 124	05	5 476 500
0·305	·333 775 294	7 832	627	1·028 146 059	0·355	·384 364 706	4 418	706	0·994 770 957
6	334 803 127	7 203	29	7 518 054	56	385 359 123	3 710	08	4 063 926
7	335 830 330	6 573	31	6 888 378	57	386 352 833	3 001	09	3 355 411
8	336 856 903	5 941	32	6 257 036	58	387 345 834	2 290	11	2 645 416
9	337 882 843	1 025 307	34	5 624 031	59	388 338 124	991 578	12	1 933 944
0·310	·338 908 150	4 671	635	1·024 989 356	0·360	·389 329 701	0 864	714	0·991 221 000
11	339 932 822	4 034	37	4 353 045	61	390 320 565	0 149	15	0 506 587
12	340 956 856	3 395	39	3 715 072	62	391 310 714	989 432	17	0·989 790 710
13	341 980 251	2 755	40	3 075 450	63	392 300 146	8 714	18	9 073 372
14	343 003 006	1 022 113	42	2 434 182	64	393 288 860	987 995	20	8 354 577
0·315	·344 025 119	1 469	644	1·021 791 274	0·365	·394 276 855	7 274	721	0·987 634 329
16	345 046 588	0 824	45	1 146 727	66	395 264 128	6 551	22	6 912 632
17	346 067 412	0 177	47	0 500 545	67	396 250 689	5 827	24	6 189 490
18	347 087 589	1 019 528	49	1·019 852 733	68	397 236 507	5 102	25	5 464 907
19	348 107 117	1 018 878	50	9 203 294	69	398 221 609	984 375	27	4 738 887
0·320	·349 125 995	8 226	652	1·018 552 231	0·370	·399 205 984	3 647	728	0·984 011 434
21	350 144 221	7 573	53	7 899 548	71	400 189 631	2 918	30	3 282 551
22	351 161 793	6 917	55	7 245 249	72	401 172 549	2 186	31	2 552 244
23	352 178 711	6 261	57	6 589 338	73	402 154 735	1 454	32	1 820 515
24	353 194 971	1 015 602	58	5 931 817	74	403 136 189	980 720	34	1 087 369
0·325	·354 210 574	4 942	660	1·015 272 691	0·375	·404 116 909	979 985	735	0·980 352 810
26	355 225 516	4 281	62	4 611 964	76	405 096 894	9 248	37	0·979 616 841
27	356 239 797	3 618	63	3 949 638	77	406 076 143	8 510	38	8 879 467
28	357 253 415	2 953	65	3 285 719	78	407 054 653	7 771	39	8 140 692
29	358 266 368	1 012 287	66	2 620 209	79	408 032 424	977 030	41	7 400 520
0·330	·359 278 655	1 619	668	1·011 953 112	0·380	·409 009 453	6 288	742	0·976 658 954
31	360 290 274	0 949	69	1 284 432	81	409 985 741	5 544	44	5 915 999
32	361 301 223	0 278	71	0 614 173	82	410 961 285	4 799	45	5 171 660
33	362 311 502	1 009 606	73	1·009 942 337	83	411 936 084	4 053	46	4 425 939
34	363 321 107	1 008 932	74	9 268 931	84	412 910 136	973 305	48	3 678 840
0·335	·364 330 039	8 256	676	1·008 593 955	0·385	·413 883 441	2 556	749	0·972 930 369
36	365 338 295	7 578	77	7 917 416	86	414 855 997	1 805	51	2 180 529
37	366 345 873	6 900	79	7 239 316	87	415 827 802	1 053	52	1 429 324
38	367 352 773	6 219	80	6 559 659	88	416 798 855	0 300	53	0 676 758
39	368 358 992	1 005 537	82	5 878 448	89	417 769 155	969 545	55	0·969 922 835
0·340	·369 364 529	4 854	684	1·005 195 689	0·390	·418 738 700	8 789	756	0·969 167 559
41	370 369 383	4 169	85	4 511 383	91	419 707 489	8 032	57	8 410 935
42	371 373 552	3 482	87	3 825 536	92	420 675 521	7 273	59	7 652 966
43	372 377 034	2 794	88	3 138 151	93	421 642 795	6 513	60	6 893 656
44	373 379 827	1 002 104	90	2 449 232	94	422 609 308	965 752	61	6 133 010
0·345	·374 381 932	1 413	691	1·001 758 782	0·395	·423 575 060	4 989	763	0·965 371 032
46	375 383 344	0 720	93	1 066 806	96	424 540 050	4 226	64	4 607 725
47	376 384 065	0 026	94	0 373 307	97	425 504 275	3 460	65	3 843 095
48	377 384 091	999 330	96	0·999 678 289	98	426 467 736	2 694	67	3 077 144
49	378 383 421	998 633	97	8 981 756	99	427 430 429	961 926	68	2 309 877

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (1) FROM $t=0$ TO $t=1{\cdot}250$.

t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$
0·400	·428 392 355	961 156	769	0·961 541 299
1	429 353 512	0 386	71	0 771 413
2	430 313 897	959 614	72	0 000 223
3	431 273 512	8 841	73	0·959 227 734
4	432 232 352	958 067	74	8 453 949
0·405	·433 190 419	7 291	776	0·957 678 873
6	434 147 710	6 514	77	6 902 511
7	435 104 224	5 736	78	6 124 865
8	436 059 959	4 956	80	5 345 941
9	437 014 915	954 175	81	4 565 742
0·410	·437 969 090	3 393	782	0·953 784 273
11	438 922 483	2 610	83	3 001 537
12	439 875 093	1 825	85	2 217 540
13	440 826 918	1 039	86	1 432 284
14	441 777 957	950 252	87	0 645 775
0·415	·442 728 209	949 464	788	0·949 858 016
16	443 677 673	8 674	90	9 069 012
17	444 626 347	7 883	91	8 278 767
18	445 574 230	7 091	92	7 487 284
19	446 521 321	946 298	93	6 694 569
0·420	447 467 618	5 503	795	0·945 900 626
21	448 413 122	4 707	96	5 105 458
22	449 357 829	3 910	97	4 309 069
23	450 301 739	3 112	98	3 511 465
24	451 244 851	942 313	99	2 712 649
0·425	·452 187 164	1 512	801	0·941 912 626
26	453 128 676	0 710	02	1 111 399
27	454 069 387	939 907	03	0 308 974
28	455 009 294	9 103	04	0·939 505 353
29	455 948 397	939 383	05	8 700 542
0·430	·456 886 695	7 491	807	0·937 894 544
31	457 824 186	6 683	08	7 087 365
32	458 760 869	5 874	09	6 279 007
33	459 696 743	5 064	10	5 469 476
34	460 631 807	934 253	11	4 658 775
0·435	461 566 060	3 440	812	0·933 846 910
36	462 499 501	2 627	14	3 033 883
37	463 432 128	1 812	15	2 219 700
38	464 363 940	0 996	16	1 404 365
39	465 294 936	930 179	17	0 587 881
0·440	·466 225 115	929 361	818	0·929 770 254
41	467 154 476	8 542	19	8 951 487
42	468 083 018	7 721	20	8 131 585
43	469 010 739	6 900	22	7 310 552
44	469 937 639	926 077	23	6 488 392
0·445	470 863 715	5 253	824	0·925 665 110
46	471 788 968	4 428	25	4 840 709
47	472 713 396	3 602	26	4 015 195
48	473 636 998	2 775	27	3 188 572
49	474 559 773	921 947	28	2 360 843

t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$
0·450	475 481 720	921 117	829	0·921 532 013
51	476 402 837	0 287	30	0 702 087
52	477 323 124	919 455	32	0·919 871 068
53	478 242 579	8 622	33	9 038 961
54	479 161 201	917 789	34	8 205 771
0·455	·480 078 990	6 954	835	0·917 371 501
56	480 995 944	6 118	36	6 536 156
57	481 912 062	5 281	37	5 699 740
58	482 827 343	4 443	38	4 862 258
59	483 741 786	913 604	39	4 023 714
0·460	·484 655 390	2 764	840	0·913 184 112
61	485 568 154	1 923	41	2 343 457
62	486 480 077	1 080	42	1 501 752
63	487 391 157	0 237	43	0 659 003
64	488 301 394	909 393	44	0·909 815 213
0·465	·489 210 787	8 548	845	0·908 970 387
66	490 119 335	7 701	46	8 124 530
67	491 027 036	6 854	47	7 277 645
68	491 933 890	6 005	48	6 429 737
69	492 839 895	905 156	49	5 580 810
0·470	·493 745 051	4 305	850	0·904 730 869
71	494 649 356	3 454	51	3 879 917
72	495 552 810	2 602	52	3 027 960
73	496 455 412	1 748	53	2 175 002
74	497 357 160	900 894	54	1 321 047
0·475	·498 258 054	0 038	855	0·900 466 099
76	499 158 092	899 182	56	0·899 610 162
77	500 057 274	8 324	57	8 753 242
78	500 955 598	7 466	58	7 895 343
79	501 853 064	896 607	59	7 036 468
0·480	·502 749 671	5 746	860	0·896 176 622
81	503 645 417	4 885	61	5 315 810
82	504 540 302	4 023	62	4 454 036
83	505 434 325	3 160	63	3 591 304
84	506 327 484	892 295	64	2 727 619
0·485	·507 219 780	1 430	865	0·891 862 985
86	508 111 210	0 564	66	0 997 406
87	509 001 774	889 697	67	0 130 888
88	509 891 471	8 829	68	0·889 263 433
89	510 780 301	887 960	69	8 395 047
0·490	·511 668 261	7 091	870	0·887 525 734
91	512 555 352	6 220	71	6 655 498
92	513 441 572	5 348	72	5 784 344
93	514 326 920	4 476	73	4 912 276
94	515 211 396	883 602	73	4 039 298
0·495	·516 094 999	2 728	874	0·883 165 416
96	516 977 727	1 853	75	2 290 632
97	517 859 580	0 977	76	1 414 952
98	518 740 556	0 100	77	0 538 381
99	519 620 656	879 222	78	0·879 660 921

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (1) FROM $t=0$ TO $t=1\cdot250$.

$t=\cdot500$] [·599

t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$	t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$
0·500	·520 499 878	878 343	879	0·878 782 579	0·550	·563 323 366	833 378	917	0·833 836 647
01	521 378 221	7 463	80	7 903 358	51	564 156 744	2 460	18	2 919 098
02	522 255 684	6 583	81	7 023 262	52	564 989 204	1 542	19	2 000 895
03	523 132 267	5 701	81	6 142 296	53	565 820 746	0 622	19	1 082 042
04	524 007 969	874 819	82	5 260 465	54	566 651 368	829 703	20	0 162 543
0·505	·524 882 788	3 936	883	0·874 377 773	0·555	·567 481 071	8 782	920	0·829 242 403
06	525 756 724	3 052	84	3 494 224	56	568 309 853	7 861	21	8 321 626
07	526 629 776	2 167	85	2 609 822	57	569 137 714	6 939	22	7 400 217
08	527 501 943	1 282	86	1 724 572	58	569 964 653	6 017	22	6 478 180
09	528 373 225	870 395	87	0 838 479	59	570 790 670	825 094	23	5 555 519
0·510	·529 243 620	869 508	887	0·869 951 547	0·560	·571 615 764	4 170	924	0·824 632 239
11	530 113 128	8 620	88	9 063 779	61	572 439 934	3 246	24	3 708 345
12	530 981 747	7 731	89	8 175 182	62	573 263 180	2 321	25	2 783 839
13	531 849 478	6 841	90	7 285 758	63	574 085 502	1 396	25	1 858 728
14	532 716 318	865 950	91	6 395 513	64	574 906 898	820 470	26	0 933 015
0·515	·533 582 268	5 059	891	0·865 504 450	0·565	·575 727 367	819 543	927	0·820 006 705
16	534 447 327	4 166	92	4 612 575	66	576 546 911	8 616	27	0·819 079 801
17	535 311 493	3 273	93	3 719 892	67	577 365 527	7 688	28	8 152 309
18	536 174 767	2 379	94	2 826 404	68	578 183 215	6 760	28	7 224 233
19	537 037 146	861 485	95	1 932 117	69	578 999 975	815 831	29	6 295 577
0·520	·537 898 630	0 589	895	0·861 037 034	0·570	·579 815 806	4 901	930	0·815 366 346
21	538 759 220	859 693	96	0 141 161	71	580 630 708	3 971	30	4 436 544
22	539 618 913	8 796	97	0·859 244 502	72	581 444 679	3 041	31	3 506 174
23	540 477 708	7 898	98	8 347 060	73	582 257 720	2 110	31	2 575 243
24	541 335 606	856 999	99	7 448 841	74	583 069 829	811 178	32	1 643 753
0·525	·542 192 606	6 100	899	0·856 549 849	0·575	·583 881 007	0 245	932	0·810 711 710
26	543 048 706	5 200	900	5 650 088	76	584 691 253	809 313	33	0·809 779 118
27	543 903 906	4 299	01	4 749 562	77	585 500 565	8 379	33	8 845 981
28	544 758 205	3 397	02	3 848 277	78	586 308 944	7 445	34	7 912 303
29	545 611 602	852 495	02	2 946 236	79	587 116 390	806 511	34	6 978 089
0·530	546 464 097	1 592	903	0·852 043 444	0·580	·587 922 900	5 576	935	0·806 043 343
31	547 315 689	0 688	04	1 139 906	81	588 728 476	4 640	36	5 108 070
32	548 166 376	849 783	05	0 235 625	82	589 533 116	3 704	36	4 172 273
33	549 016 160	8 878	05	0·849 330 606	83	590 336 821	2 768	37	3 235 958
34	549 865 037	847 972	06	8 424 853	84	591 139 588	801 831	37	2 299 129
0·535	·550 713 009	7 065	907	0·847 518 372	0·585	·591 941 419	0 893	938	0·801 361 789
36	551 560 074	6 157	08	6 611 165	86	592 742 311	799 955	38	0 423 943
37	552 406 231	5 249	08	5 703 239	77	593 542 266	9 016	39	0·799 485 597
38	553 251 480	4 340	09	4 794 596	78	594 341 283	8 077	39	8 546 753
39	554 095 820	843 430	10	3 885 242	79	595 139 360	797 138	40	7 607 416
0·540	·554 939 250	2 520	910	0·842 975 181	0·590	·595 936 497	6 197	940	0·796 667 591
41	555 781 770	1 609	11	2 064 417	91	596 732 695	5 257	41	5 727 282
42	556 623 379	0 697	12	1 152 955	92	597 527 952	4316	41	4 786 493
43	557 464 076	839 784	13	0 240 799	93	598 322 268	3 374	42	3 845 229
44	558 303 860	838 871	13	0·839 327 954	94	599 115 642	792 432	42	2 903 494
0·545	·559 142 732	7 957	914	0·838 414 423	0·595	·599 908 074	1 490	942	0·791 961 292
46	559 980 689	7 043	15	7 500 212	96	600 699 564	0 547	43	1 018 627
47	560 817 732	6 128	15	6 585 324	97	601 490 111	789 604	43	0 075 505
48	561 653 859	5 212	16	5 669 765	98	602 279 715	8 660	44	0·789 131 928
49	562 489 071	834 295	17	4 753 538	99	603 068 375	787 716	44	0·788 187 903

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (1) FROM $t=0$ TO $t=1·250$.

$t=$ ·600] [to ·699

t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$
0·600	·603 856 091	786 771	945	0·787 243 432
01	4 642 862	5 826	45	6 298 520
02	5 428 688	4 880	46	5 353 172
03	6 213 568	3 934	46	4 407 391
04	6 997 502	782 988	46	3 461 182
0·605	·607 780 490	2 041	947	0·782 514 550
06	8 562 531	1 094	47	1 567 499
07	9 343 625	0 146	48	0 620 032
08	·610 123 771	779 198	48	0 779 672 155
09	0 902 969	778 250	48	8 723 871
0·610	·611 681 219	7 301	949	0·777 775 185
11	2 458 520	6 351	49	6 826 101
12	3 234 871	5 402	50	5 876 623
13	4 010 273	4 452	50	4 926 756
14	4 784 724	773 501	50	3 976 504
0·615	·615 558 226	2 550	951	0·773 025 871
16	6 330 776	1 599	51	2 074 862
17	7 102 375	0 648	52	1 123 480
18	7 873 023	769 696	52	0 171 731
19	8 642 718	768 743	52	0·769 219 617
0·620	·619 411 462	7 791	953	0·768 267 144
21	·620 179 253	6 838	53	7 314 316
22	0 946 090	5 884	53	6 361 137
23	1 711 975	4 931	54	5 407 611
24	2 476 906	763 977	54	4 453 742
0·625	·623 240 882	3 022	954	0·763 499 536
26	4 003 904	2 068	55	2 544 995
27	4 765 972	1 113	55	1 590 125
28	5 527 085	0 157	55	0 634 928
29	6 287 242	759 202	56	0·759 679 411
0·630	·627 046 443	8 246	956	0·758 723 576
31	7 804 689	7 289	56	7 767 429
32	8 561 978	6 333	57	6 810 973
33	9 318 311	5 376	57	5 854 212
34	·630 073 686	754 418	57	4 897 151
0·635	·630 828 105	3 461	958	0·753 939 794
36	1 581 566	2 503	58	2 982 146
37	2 334 069	1 545	58	2 024 209
38	3 085 614	0 587	58	1 065 989
39	3 836 201	749 628	59	0 107 490
0·640	·634 585 829	8 669	959	0·749 148 716
41	5 334 498	7 710	59	8 189 671
42	6 082 208	6 751	59	7 230 359
43	6 828 959	5 791	60	6 270 785
44	7 574 750	744 831	60	5 310 952
0·645	·638 319 581	3 871	960	0·744 350 865
46	9 063 452	2 910	60	3 390 528
47	9 806 362	1 950	61	2 429 945
48	·640 548 311	0 989	61	1 469 121
49	1 289 300	740 027	61	0 508 059

t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$
0·650	·642 029 327	739 066	961	0·739 546 763
51	2 768 393	8 104	62	8 585 239
52	3 506 498	7 143	62	7 623 489
53	4 243 640	6 180	62	6 661 518
54	4 979 821	735 218	62	5 699 330
0·655	·645 715 039	4 256	963	0·734 736 930
56	6 449 295	3 293	63	3 774 321
57	7 182 587	2 330	63	2 811 507
58	7 914 917	1 367	63	1 848 494
59	8 646 284	730 404	63	0 885 284
0·660	·649 376 688	729 440	963	0·729 921 881
61	·650 106 128	8 476	64	8 958 291
62	0 834 604	7 513	64	7 994 517
63	1 562 117	6 549	64	7 030 563
64	2 288 666	725 584	64	6 066 433
0·665	·653 014 250	4 620	964	0·725 102 132
66	3 738 870	3 655	65	4 137 663
67	4 462 525	2 691	65	3 173 031
68	5 185 216	1 726	65	2 208 239
69	5 906 942	720 761	65	1 243 292
0·670	·656 627 702	719 796	965	0·720 278 193
71	7 347 498	8 830	65	0·719 312 947
72	8 066 328	7 865	65	8 347 558
73	8 784 193	6 899	66	7 382 030
74	9 501 092	715 933	66	6 416 367
0·675	·660 217 026	4 968	966	0·715 450 573
76	0 931 993	4 002	66	4 484 652
77	1 645 995	3 036	66	3 518 608
78	2 359 030	2 069	66	2 552 445
79	3 071 100	711 103	66	1 586 167
0·680	·663 782 203	0 137	966	0·710 619 778
81	4 492 339	709 170	67	0·709 653 283
82	5 201 509	8 203	67	8 686 684
83	5 909 713	7 237	67	7 719 987
84	6 616 949	706 270	67	6 753 195
0·685	·667 323 219	5 303	967	0·705 786 311
86	8 028 522	4 336	67	4 819 341
87	8 732 858	3 369	67	3 852 288
88	9 436 226	2 402	67	2 885 157
89	·670 138 628	701 434	67	1 917 950
0·690	·670 840 062	0 467	967	0·700 950 672
91	1 540 529	699 500	67	0·699 983 327
92	2 240 029	8 532	67	9 015 919
93	2 938 561	7 565	67	8 048 453
94	3 636 126	696 597	68	7 080 930
0·695	·674 332 723	5 630	968	0·696 113 357
96	5 028 352	4 662	68	5 145 737
97	5 723 014	3 694	68	4 178 073
98	6 416 709	2 727	68	3 210 370
99	7 109 435	691 759	68	2 242 631

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (I) FROM $t=0$ TO $t=1{\cdot}250$.

t. ·700] [to ·799

t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$	t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$
0·700	·677 801 194	690 791	968	0·691 274 860	0·750	·711 155 634	642 449	964	0·642 931 069
01	8 491 985	689 823	68	0 307 062	51	1 798 083	1 485	64	1 966 754
02	9 181 808	8 855	68	0·689 339 241	52	2 439 567	0 521	64	1 002 602
03	9 870 663	7 887	68	8 371 399	53	3 080 088	639 557	64	0 038 619
04	·680 558 551	686 920	68	7 403 542	54	3 719 645	638 593	64	0·639 074 807
0·705	·681 245 470	5 952	968	0·686 435 672	0·755	·714 358 237	7 629	964	0·638 111 171
06	1 931 422	4 984	68	5 467 794	56	4 995 867	6 666	63	7 147 713
07	2 616 406	4 016	68	4 499 912	57	5 632 533	5 703	63	6 184 437
08	3 300 422	3 048	68	3 532 030	58	6 268 236	4 740	63	5 221 347
09	3 983 470	682 080	68	2 564 151	59	6 902 976	633 777	63	4 258 447
0·710	·684 665 550	1 112	968	0·681 596 279	0·760	·717 536 753	2 814	963	0·633 295 740
11	5 346 663	0 144	68	0 628 419	61	8 169 567	1 852	62	2 333 229
12	6 026 807	679 177	68	0·679 660 573	62	8 801 419	0 890	62	1 370 919
13	6 705 984	8 209	68	8 692 747	63	9 432 309	629 928	62	0 408 812
14	7 384 193	677 241	68	7 724 943	64	·720 062 237	628 966	62	0·629 446 912
0·715	·688 061 434	6 273	968	0·676 757 166	0·765	·720 691 203	8 004	962	0·628 485 223
16	8 737 707	5 306	68	5 789 419	66	1 319 208	7 043	61	7 523 749
17	9 413 012	4 338	68	4 821 706	67	1 946 251	6 082	61	6 562 492
18	·690 087 350	3 370	68	3 854 031	68	2 572 333	5 121	61	5 601 456
19	0 760 721	672 403	68	2 886 398	69	3 197 454	624 160	61	4 640 645
0·720	·691 433 123	1 435	968	0·671 918 811	0·770	·723 821 614	3 200	960	0·623 680 063
21	2 104 558	0 468	68	0 951 274	71	4 444 814	2 240	60	2 719 712
22	2 775 026	669 500	67	0·669 983 789	72	5 067 053	1 280	60	1 759 596
23	3 444 526	8 533	67	9 016 362	73	5 688 333	0 320	60	0 799 719
24	4 113 058	667 565	67	8 048 995	74	6 308 653	619 360	60	0·619 840 085
0·725	·694 780 624	6 598	967	0·667 081 693	0·775	·726 928 013	8 401	959	0·618 880 695
26	5 447 222	5 631	67	6 114 459	76	7 546 414	7 442	59	7 921 556
27	6 112 853	4 664	67	5 147 298	77	8 163 857	6 483	59	6 962 668
28	6 777 516	3 697	67	4 180 212	78	8 780 340	5 525	59	6 004 037
29	7 441 213	662 730	67	3 213 206	79	9 395 865	614 567	58	5 045 665
0·730	·698 103 943	1 763	966	0·662 246 284	0·780	·730 010 431	3 609	958	0·614 087 556
31	8 765 706	0 796	67	1 279 448	81	0 624 040	2 651	58	3 129 713
32	9 426 502	659 829	67	0 312 704	82	1 236 691	1 693	57	2 172 139
33	·700 086 331	8 863	67	0·659 346 054	83	1 848 384	0 736	57	1 214 839
34	0 745 194	657 896	67	8 379 503	84	2 459 121	609 779	57	0 257 816
0·735	·701 403 090	6 930	966	0·657 413 053	0·785	·733 068 900	8 823	957	0·609 301 072
36	2 060 020	5 964	66	6 446 709	86	3 677 723	7 867	56	8 344 611
37	2 715 984	4 997	66	5 480 475	87	4 285 589	6 910	56	7 388 438
38	3 370 981	4 031	66	4 514 354	88	4 892 500	5 955	56	6 432 554
39	4 025 012	653 065	66	3 548 350	89	5 498 455	604 999	55	5 476 963
0·740	·704 678 078	2 100	966	0·652 582 466	0·790	·736 103 454	4 044	955	0·604 521 670
41	5 330 177	1 134	66	1 616 707	91	6 707 498	3 089	55	3 566 676
42	5 981 311	0 168	66	0 651 076	92	7 310 587	2 135	55	2 611 986
43	6 631 480	649 203	65	0·649 685 576	93	7 912 722	1 181	54	1 657 602
44	7 280 683	648 238	65	8 720 211	94	8 513 903	600 227	54	0 703 529
0·745	·707 928 920	7 272	965	0·647 754 986	0·795	·739 114 129	599 273	954	0·599 749 769
46	8 576 193	6 307	65	6 789 903	96	9 713 402	8 320	53	8 796 326
47	9 222 500	5 343	65	5 824 966	97	·740 311 722	7 367	53	7 843 203
48	9 867 843	4 378	65	4 860 179	98	0 909 089	6 414	53	6 890 403
49	·710 512 220	643 413	65	3 895 546	99	1 505 503	595 462	52	5 937 930

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2} dt$. (1) FROM $t=0$ TO $t=1.250$.

t = ·800] [·899

t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$	t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$
0·800	·742 100 965	594 510	952	0·594 985 786	0·850	·770 668 058	547 404	931	0·547 869 717
01	2 695 475	3 558	52	4 033 976	51	1 215 462	6 473	31	6 938 583
02	3 289 033	2 607	51	3 082 502	52	1 761 935	5 543	30	6 007 939
03	3 881 640	1 656	51	2 131 369	53	2 307 478	4 613	30	5 077 789
04	4 473 296	590 705	51	1 180 578	54	2 852 091	543 684	29	0·544 148 135
0·805	·745 064 001	589 755	950	0·590 230 133	0·855	·773 395 774	2 755	929	0·543 218 980
06	5 653 756	8 805	50	0·589 280 038	56	3 938 529	1 826	28	2 290 327
07	6 242 561	7 856	50	8 330 295	57	4 480 355	0 898	28	1 362 179
08	6 830 417	6 906	49	7 380 909	58	5 021 253	539 971	27	0 434 538
09	7 417 323	585 958	49	6 431 881	59	5 561 224	539 044	27	0·539 507 408
0·810	·748 003 281	5 009	948	0·585 483 216	0·860	·776 100 268	8 118	926	0·538 580 792
11	8 588 290	4 061	48	4 534 917	61	6 638 386	7 192	26	7 654 691
12	9 172 351	3 113	48	3 586 986	62	7 175 578	6 267	25	6 729 110
13	9 755 464	2 166	47	2 639 427	63	7 711 844	5 342	25	5 804 050
14	·750 337 630	581 219	47	1 692 243	64	8 247 186	534 417	24	4 879 515
0·815	·750 918 848	0 272	947	0·580 745 438	0·865	·778 781 604	3 494	924	0·533 955 508
16	1 499 121	579 326	46	0·579 799 014	66	9 315 097	2 571	23	3 032 030
17	2 078 446	8 380	46	8 852 975	67	9 847 668	1 648	23	2 109 086
18	2 656 827	7 435	45	7 907 324	68	·780 379 316	0 726	22	1 186 677
19	3 234 261	576 490	45	6 962 064	69	0 910 041	529 804	22	0 264 806
0·820	·753 810 751	5 545	945	0·576 017 197	0·870	·781 439 845	8 883	921	0·529 343 477
21	4 386 296	4 601	44	5 072 728	71	1 968 729	7 963	21	8 422 692
22	4 960 896	3 657	44	4 128 659	72	2 496 691	7 043	20	7 502 454
23	5 534 553	2 713	43	3 184 994	73	3 023 734	6 123	19	6 582 764
24	6 107 267	571 770	43	2 241 736	74	3 549 857	525 204	19	5 663 627
0·825	·756 679 037	0 828	943	0·571 298 886	0·875	·784 075 061	4 286	918	0·524 745 045
26	7 249 864	569 885	42	0 356 450	76	4 599 347	3 368	18	3 827 021
27	7 819 750	8 944	42	0·569 414 430	77	5 122 715	2 451	17	2 909 556
28	8 388 693	8 002	41	8 472 828	78	5 645 166	1 534	17	1 992 655
29	8 956 696	567 061	41	7 531 649	79	6 166 701	520 618	16	1 076 319
0·830	·759 523 757	6 121	941	0·566 590 894	0·880	·786 687 319	519 703	915	0·520 160 551
31	·760 089 878	5 181	40	5 650 568	81	7 207 022	8 788	15	0·519 245 355
32	0 655 058	4 241	40	4 710 673	82	7 725 810	7 874	14	8 330 732
33	1 219 299	3 302	39	3 771 212	83	8 243 684	6 960	14	7 416 685
34	1 782 601	562 363	39	2 832 188	84	8 760 644	516 047	13	6 503 217
0·835	·762 344 964	1 424	938	0·561 893 605	0·885	·789 276 690	5 134	913	0·515 590 330
36	2 906 388	0 487	38	0 955 465	86	9 791 825	4 222	12	4 678 028
37	3 466 875	559 549	37	0 017 771	87	·790 306 047	3 311	11	3 766 312
38	4 026 424	8 612	37	0·559 080 526	88	0 819 357	2 400	11	2 855 186
39	4 585 036	557 676	37	8 143 734	89	1 331 757	511 490	10	1 944 651
0·840	·765 142 712	6 739	936	0·557 207 397	0·890	·791 843 247	0 580	910	0·511 034 712
41	5 699 451	5 804	36	6 271 518	91	2 353 827	509 671	09	0 125 369
42	6 255 255	4 869	35	5 336 100	92	2 863 498	8 763	08	0·509 216 626
43	6 810 123	3 934	35	4 401 147	93	3 372 260	7 855	08	8 308 485
44	7 364 057	553 000	34	3 466 661	94	3 880 115	506 947	07	7 400 949
0·845	·767 917 057	2 066	934	0·552 532 644	0·895	·794 387 062	6 041	907	0·506 494 020
46	8 469 123	1 133	33	1 599 101	96	4 893 103	5 135	06	5 587 701
47	9 020 255	0 200	33	0 666 034	97	5 398 238	4 229	05	4 681 994
48	9 570 455	549 267	32	0·549 733 446	98	5 902 467	3 325	05	3 776 903
49	·770 119 722	548 335	32	8 801 339	99	6 405 792	502 420	04	2 872 428

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (1) FROM $t=0$ TO $t=1·250$.

$t=·900$] [·999

t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$	t	H	Δ +	Δ −	$\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}$
0·900	·796 908 212	501 517	904	0·501 968 574	0·950	·820 890 807	457 185	869	0·457 619 255
01	7 409 729	0 614	03	1 065 342	51	1 347 992	6 316	69	6 750 147
02	7 910 343	499 712	02	0 162 736	52	1 804 308	5 448	68	5 881 778
03	8 410 055	8 810	02	0·499 260 756	53	2 259 756	4 581	67	5 014 150
04	8 908 865	497 909	01	8 359 407	54	2 714 336	453 714	67	4 147 265
0·905	·799 406 774	7 009	900	0·497 458 689	0·955	·823 168 050	2 848	866	0·453 281 124
06	9 903 783	6 109	00	6 558 607	56	3 620 899	1 983	65	2 415 731
07	·800 399 891	5 210	899	5 659 162	57	4 072 882	1 119	64	1 551 087
08	0 895 101	4 311	98	4 760 356	58	4 524 001	0 256	64	0 687 194
09	1 389 412	493 413	98	3 862 193	59	4 974 257	449 393	63	0·449 824 055
0·910	·801 882 826	2 516	897	0·492 964 674	0·960	·825 423 650	8 531	862	0·448 961 670
11	2 375 342	1 620	97	2 067 802	61	5 872 180	7 670	61	8 100 042
12	2 866 962	0 724	96	1 171 580	62	6 319 850	6 809	60	7 239 174
13	3 357 685	489 828	95	0 276 009	63	6 766 659	5 949	60	6 379 067
14	3 847 514	488 934	95	0·489 381 093	64	7 212 608	445 090	59	5 519 723
0·915	·804 336 448	8 040	894	0·488 486 833	0·965	·827 657 699	4 232	858	0·444 661 143
16	4 824 488	7 147	93	7 593 232	66	8 101 931	3 375	57	3 803 331
17	5 311 634	6 254	93	6 700 292	67	8 545 306	2 518	57	2 946 288
18	5 797 889	5 362	92	5 808 016	68	8 987 824	1 662	56	2 090 016
19	6 283 251	484 471	91	4 916 406	69	9 429 486	440 807	55	1 234 516
0·920	·806 767 722	3 580	891	0·484 025 464	0·970	·829 870 293	439 953	854	0·440 379 791
21	7 251 302	2 690	90	3 135 193	71	·830 310 246	9 099	54	0·439 525 843
22	7 733 992	1 801	89	2 245 595	72	0 749 345	8 246	53	8 672 674
23	8 215 793	0 912	89	1 356 672	73	1 187 591	7 394	52	7 820 284
24	8 696 706	480 025	88	0 468 427	74	1 624 986	436 543	51	6 968 678
0·925	·809 176 730	479 137	887	0·479 580 861	0·975	·832 061 529	5 693	850	0·436 117 855
26	9 655 868	8 251	87	8 693 978	76	2 497 222	4 843	50	5 267 819
27	·810 134 119	7 365	86	7 807 779	77	2 932 065	3 994	49	4 418 570
28	0 611 483	6 480	85	6 922 268	78	3 366 059	3 146	48	3 570 111
29	1 087 963	475 595	84	6 037 445	79	3 799 205	432 299	47	2 722 444
0·930	·811 563 559	4 712	884	0·475 153 313	0·980	·834 231 504	1 453	846	0·431 875 571
31	2 038 270	3 828	83	4 269 875	81	4 662 957	0 607	46	1 029 493
32	2 512 099	2 946	82	3 387 133	82	5 093 564	429 762	45	0 184 212
33	2 985 045	2 064	82	2 505 089	83	5 523 326	8 918	44	0·429 339 730
34	3 457 109	471 183	81	1 623 745	84	5 952 243	428 075	43	8 496 049
0·935	813 928 292	0 303	880	0·470 743 103	0·985	·836 380 318	7 232	842	0·427 653 170
36	4 398 595	469 423	80	0·469 863 166	86	6 807 550	6 390	42	6 811 096
37	4 868 019	8 545	79	8 983 936	87	7 233 940	5 541	41	5 969 828
38	5 336 564	7 666	78	8 105 415	88	7 659 490	4 709	40	5 129 367
39	5 804 230	466 789	77	7 227 605	89	8 084 199	423 870	39	4 289 717
0·940	816 271 019	5 912	877	0·466 350 509	0·990	·838 508 070	3 032	838	0·423 450 878
41	6 736 931	5 036	76	5 474 128	91	8 931 101	2 194	38	2 612 852
42	7 201 967	4 161	75	4 598 465	92	9 353 296	1 357	37	1 775 641
43	7 666 128	3 286	75	3 723 522	93	9 774 653	0 521	36	0 939 247
44	8 129 415	462 412	74	2 849 301	94	·840 195 174	419 686	35	0 103 671
0·945	·818 591 827	1 539	873	0·461 975 804	0·995	·840 614 861	8 852	834	0·419 268 915
46	9 053 367	0 667	72	1 103 033	96	1 033 712	8 018	34	8 434 981
47	9 514 034	459 795	72	0 230 991	97	1 451 731	7 186	33	7 601 871
48	9 973 829	8 924	71	0·459 359 679	98	1 868 916	6 354	32	6 769 586
49	·820 432 753	458 054	70	8 489 099	99	2 285 270	415 523	31	5 938 127

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (1) FROM $t=0$ TO $t=1·250$.

$t = 1·000$] [1·099

t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$	t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$
1·000	·842 700 793	414 693	830	0·415 107 497	1·050	·862 436 106	374 274	787	0·374 666 957
01	3 115 485	3 863	29	4 277 698	51	2 810 380	3 488	86	3 880 608
02	3 529 349	3 035	29	3 448 730	52	3 183 868	2 703	85	3 095 163
03	3 942 383	2 207	28	2 620 596	53	3 556 570	1 919	84	2 310 624
04	4 354 590	411 380	27	1 793 297	54	3 928 489	371 136	83	1 526 991
1·005	·844 765 970	0 554	826	0·410 966 835	1·055	·864 299 625	0 353	782	0·370 744 267
06	5 176 524	409 729	25	0 141 211	56	4 669 978	369 572	81	0 369 962 451
07	5 586 253	8 904	24	0·409 316 427	57	5 039 550	8 791	80	9 181 546
08	5 995 157	8 081	24	8 492 485	58	5 408 341	8 012	80	8 401 552
09	6 403 238	407 258	23	7 669 386	59	5 776 353	367 233	79	7 622 471
1·010	·846 810 496	6 436	822	0·406 847 132	1·060	·866 143 587	6 456	778	0·366 844 303
11	7 216 933	5 615	21	6 025 724	61	6 510 042	5 679	77	6 067 051
12	7 622 548	4 795	20	5 205 164	62	6 875 721	4 903	76	5 290 715
13	8 027 343	3 976	19	4 385 454	63	7 240 624	4 128	75	4 515 297
14	8 431 319	403 158	18	3 566 595	64	7 604 752	363 354	74	3 740 797
1·015	·848 834 477	2 340	818	0·402 748 588	1·065	·867 968 106	2 581	773	0·362 967 216
16	9 236 817	1 523	17	1 931 436	66	8 330 687	1 809	72	2 194 557
17	9 638 340	0 707	16	1 115 139	67	8 692 495	1 037	71	1 422 819
18	·850 039 047	399 892	15	0 299 700	68	9 053 533	0 267	70	0 652 004
19	0 438 940	399 078	14	0·399 485 119	69	9 413 800	359 498	69	0·359 882 114
1·020	·850 838 018	8 265	813	0·398 671 399	1·070	·869 773 297	8 729	769	0·359 113 149
21	1 236 283	7 452	12	7 858 541	71	870 132 026	7 961	68	8 345 110
22	1 633 735	6 641	12	7 046 545	72	0 489 988	7 195	67	7 577 999
23	2 030 376	5 830	11	6 235 415	73	0 847 183	6 429	66	6 811 816
24	2 426 206	395 020	10	5 425 151	74	1 203 612	355 664	65	6 046 563
1·025	·852 821 227	4 211	809	0·394 615 754	1·075	·871 559 276	4 900	764	0·355 282 240
26	3 215 438	3 403	08	3 807 226	76	1 914 176	4 138	63	4 518 850
27	3 608 841	2 596	07	2 999 570	77	2 268 314	3 376	62	3 756 392
28	4 001 437	1 790	06	2 192 785	78	2 621 690	2 614	61	2 994 867
29	4 393 227	390 984	05	1 386 873	79	2 974 304	351 854	60	2 234 278
1·030	·854 784 211	0 180	805	0.390 581 837	1·080	·873 326 158	1 095	759	0·351 474 625
31	5 174 391	389 376	04	0·389 777 677	81	3 677 254	0 337	58	0 715 908
32	5 563 767	8 573	03	8 974 394	82	4 027 591	349 580	57	0·349 958 129
33	5 952 340	7 771	02	8 171 991	83	4 377 170	8 823	56	9 201 289
34	6 340 111	386 970	01	7 370 468	84	4 725 993	348 068	55	8 445 390
1·035	·856 727 081	6 170	800	0·386 569 827	1·085	·875 074 061	7 313	754	0·347 690 431
36	7 113 251	5 371	799	5 770 069	86	5 421 375	6 560	54	6 936 413
37	7 498 622	4 572	98	4 971 196	87	5 767 934	5 807	53	6 183 339
38	7 883 194	3 775	98	4 173 209	88	6 113 742	5 056	52	5 431 209
39	8 266 969	382 978	97	3 376 110	89	6 458 797	344 305	51	4 680 023
1·040	·858 649 947	2 182	796	0·382 579 899	1·090	·876 803 102	3 555	750	0·343 929 783
41	9 032 129	1 387	95	1 784 578	91	7 146 657	2 806	49	3 180 489
42	9 413 516	0 593	94	0 990 148	92	7 489 463	2 058	48	2 432 143
43	9 794 109	379 800	93	0 196 611	93	7 831 522	1 311	47	1 684 746
44	·860 173 909	79 008	92	0·379 403 968	94	8 172 833	340 565	46	0 938 298
1·045	·860 552 918	8 217	791	0·378 612 221	1·095	·878 513 399	339 820	745	0·340 192 800
46	0 931 134	7 426	90	7 821 370	96	8 853 219	9 076	44	0·339 448 254
47	1 308 561	6 637	90	7 031 416	97	9 192 295	8 333	43	8 704 659
48	1 685 197	5 848	89	6 242 362	98	9 530 629	7 591	42	7 962 018
49	2 061 046	375 061	88	5 454 209	99	9 868 220	336 850	41	7 220 331

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (I) FROM $t=0$ TO $t=1\cdot250$.

$t = 1\cdot100$] [1·199

t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$	t	H	Δ +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$
1·100	·880 205 070	336 110	740	0·336 479 598	1·150	·896 123 843	300 332	692	0·300 677 276
01	0 541 179	5 370	39	5 739 821	51	6 424 175	299 641	91	0·299 986 213
02	0 876 550	4 632	38	5 001 000	52	6 723 816	8 952	90	9 296 140
03	1 211 182	3 895	37	4 263 136	53	7 022 767	8 263	89	8 607 056
04	1 545 076	333 158	36	3 526 231	54	7 321 030	297 575	88	7 918 964
1·105	·881 878 234	2 423	735	0·332 790 285	1·155	·897 618 605	6 889	687	0·297 231 863
06	2 210 657	1 688	35	2 055 298	56	7 915 494	6 203	86	6 545 753
07	2 542 345	0 955	34	1 321 273	57	8 211 697	5 518	85	5 860 635
08	2 873 300	0 222	33	0 588 208	58	8 507 216	4 835	84	5 176 510
09	3 203 522	329 490	32	0·329 856 106	59	8 802 051	294 152	83	4 493 378
1·110	·883 533 012	8 760	731	0·329 124 967	1·160	·899 096 203	3 471	682	0·293 811 239
11	3 861 772	8 030	30	8 394 791	61	9 389 673	2 790	81	3 130 094
12	4 189 802	7 301	29	7 665 581	62	9 682 463	2 110	80	2 449 943
13	4 517 104	6 574	28	6 937 335	63	9 974 574	1 432	79	1 770 787
14	4 843 677	325 847	27	6 210 056	64	·900 266 005	290 754	78	1 092 625
1·115	·885 169 524	5 121	726	0·325 483 743	1·165	·900 556 759	0 077	677	0·290 415 459
16	5 494 645	4 396	25	4 758 398	66	0 846 837	289 402	76	0·289 739 289
17	5 819 041	3 672	24	4 034 022	67	1 136 238	8 727	75	9 064 116
18	6 142 713	2 949	23	3 310 615	68	1 424 965	8 053	74	8 389 938
19	6 465 663	322 227	22	2 588 177	69	1 713 018	287 381	73	7 716 758
1·120	·886 787 890	1 506	721	0·321 866 710	1·170	·902 000 399	6 709	672	0·287 044 575
21	7 109 397	0 786	20	1 146 215	71	2 287 108	6 038	71	6 373 389
22	7 430 183	0 067	19	0 426 691	72	2 573 146	5 369	70	5 703 202
23	7 750 250	319 349	18	0·319 708 140	73	2 858 515	4 700	69	5 034 013
24	8 069 600	318 632	17	8 990 562	74	3 143 214	284 032	68	4 365 822
1·125	·888 388 232	7 916	716	0·318 273 959	1·175	·903 427 247	3 365	667	0·283 698 631
26	8 706 148	7 201	15	7 558 330	76	3 710 612	2 700	66	3 032 439
27	9 023 349	6 487	14	6 843 676	77	3 993 312	2 035	65	2 367 247
28	9 339 835	5 774	13	6 129 998	78	4 275 347	1 371	64	1 703 054
29	9 655 609	315 061	12	5 417 298	79	4 556 718	280 709	63	1 039 862
1·130	·889 970 670	4 350	711	0·314 705 574	1·180	·904 837 427	0 047	662	0·280 377 670
31	·890 285 020	3 640	10	3 994 829	81	5 117 474	279 386	61	0·279 716 479
32	0 598 660	2 931	09	3 285 062	82	5 396 860	8 727	60	9 056 290
33	0 911 591	2 222	08	2 576 274	83	5 675 587	8 068	59	8 397 101
34	1 223 813	311 515	07	1 868 466	84	5 953 655	277 410	58	7 738 915
1·135	·891 535 328	0 809	706	0·311 161 639	1·185	·906 231 065	6 754	657	0·277 081 730
36	1 846 137	0 103	05	0 455 793	86	6 507 819	6 098	56	6 425 547
37	2 156 240	309 399	04	0·309 750 928	87	6 783 916	5 443	55	5 770 367
38	2 465 639	8 696	03	9 047 046	88	7 059 360	4 790	54	5 116 190
39	2 774 335	307 993	02	8 344 146	89	7 334 149	274 137	53	4 463 015
1·140	·893 082 328	7 292	701	0·307 642 230	1·190	·907 608 286	3 485	652	0·273 810 844
41	3 389 619	6 591	00	6 941 298	91	7 881 771	2 835	51	3 159 676
42	3 696 211	5 892	699	6 241 350	92	8 154 606	2 185	50	2 509 511
43	4 002 102	5 193	98	5 542 387	93	8 426 791	1 536	49	1 860 350
44	4 307 296	304 496	97	4 844 410	94	8 698 327	270 889	48	1 212 194
1·145	·894 611 791	3 799	696	0·304 147 420	1·195	·908 969 215	0 242	647	0·270 565 041
46	4 915 591	3 104	96	3 451 415	96	9 239 457	269 596	46	0·269 918 893
47	5 218 695	2 409	95	2 756 398	97	9 509 053	8 952	45	9 273 749
48	5 521 104	1 716	94	2 062 369	98	9 778 005	8 308	44	8 629 610
49	5 822 820	301 023	93	1 369 328	99	·910 046 313	267 665	43	7 986 476

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (1) FROM $t=0$ TO $t=1\cdot250$.

1·200] 1·249] [1·25 [6·0

t	H	Δ_1 +	Δ_2 −	$\frac{2}{\sqrt{\pi}}e^{-t^2}$	t	$\frac{2}{\sqrt{\pi}}e^{-t^2}$	t	$\frac{2}{\sqrt{\pi}}e^{-t^2}$
1·200	·910 313 978	267 024	642	0·267 344 347	1·25	·236 521 122 447 291	2·00	·020 666 985 354 092
1	0 581 002	6 383	1	6 703 223	1·26	·230 658 328 140 766	2·02	·019 070 402 324 130
2	0 847 385	5 743	0	6 063 105	1·27	·224 895 874 843 793	2·04	·017 583 087 747 389
3	1 113 128	5 105	639	5 423 992	1·28	·219 233 531 734 808	2·06	·016 198 805 688 967
4	1 378 233	264 467	8	4 785 885	1·29	·213 671 014 478 186	2·08	·014 911 571 415 508
1·205	·911 642 701	3 831	637	0·264 148 783	1·30	·208 207 986 796 070	2·10	·013 715 649 999 807
6	1 906 531	3 195	6	3 512 687	1·31	·202 844 062 051 747	2·12	·012 605 554 077 274
7	2 169 726	2 560	5	2 877 598	1·32	·197 578 804 842 403	2·15	·011 089 930 302 398
8	2 432 287	1 927	4	2 243 514	1·33	·192 411 732 599 137	2·17	·010 171 986 461 662
9	2 694 214	261 294	3	1 610 437	1·34	·187 342 317 192 141	2·20	·008 922 155 064 916
1·210	·912 955 508	0 663	632	0·260 978 366	1·35	·182 369 986 539 023	2·25	·007 142 319 022 018
11	3 216 171	0 032	1	0 347 302	1·36	·177 494 126 214 278	2·30	·005 689 017 242 525
12	3 476 203	259 403	0	0·259 717 244	1·37	·172 714 081 057 979	2·35	·004 508 829 189 593
13	3 735 606	8 774	629	9 088 193	1·38	·168 029 156 781 793	2·40	·003 555 648 680 878
14	3 994 380	258 147	8	8 460 148	1·39	·163 438 621 570 507	2·45	·002 789 988 619 011
1·215	·914 252 526	7 520	627	0·257 833 110	1·40	·158 941 707 677 278	2·50	·002 178 284 230 353
16	4 510 046	6 894	6	7 207 079	1·41	·154 537 613 010 895	2·55	·001 692 213 637 679
17	4 766 941	6 270	5	6 582 055	1·42	·150 225 502 713 389	2·60	·001 308 050 049 723
18	5 023 211	5 646	4	5 958 038	1·43	·146 004 510 726 399	2·65	·001 006 055 779 156
19	5 278 857	255 024	3	5 335 028	1·44	·141 873 741 344 739	2·70	·000 769 924 759 855
1·220	·915 533 881	4 402	621	0·254 713 024	1·45	·137 832 270 755 693	2·75	·000 586 277 247 094
21	5 788 283	3 782	0	4 092 028	1·46	·133 879 148 562 625	2·80	·000 444 207 944 206
22	6 042 065	3 162	619	3 472 039	1·47	·130 013 399 291 529	2·85	·000 334 886 877 468
23	6 295 228	2 544	8	2 853 057	1·48	·126 234 023 879 239	2·90	·000 251 210 892 521
24	6 547 772	251 927	7	2 235 082	1·49	·122 540 001 142 057	2·95	·000 187 502 615 679
1·225	·916 799 698	1 310	616	0·251 618 114	1·50	·118 930 289 223 629	3·00	·000 139 253 051 983
26	7 051 008	0 695	5	1 002 153	1·52	·111 959 535 587 539	3·1	·000 075 663 266 797
27	7 301 703	0 080	4	0 387 199	1·54	·105 313 068 275 229	3·2	·000 040 297 635 533
28	7 551 783	249 467	3	0·249 773 253	1·56	·098 981 950 627 349	3·3	·000 021 037 210 443
29	7 801 250	248 854	2	9 160 313	1·58	·092 957 046 104 774	3·4	·000 010 764 921 037
1·230	·918 050 104	8 243	611	0·248 548 381	1·60	·087 229 058 633 945	3·5	* ·(5)5 399 426 777 385
31	8 298 347	7 632	0	7 937 455	1·62	·081 788 571 130 589	3·6	·(5)2 654 596 844 717
32	8 545 979	7 023	609	7 327 536	1·64	·076 626 082 133 553	3·7	·(5)1 279 274 084 534
33	8 793 002	6 415	8	6 718 625	1·66	·071 732 040 494 964	3·8	·(5)0 604 286 289 322
34	9 039 417	245 807	7	6 110 720	1·68	·067 096 878 086 755	3·9	·(5)0 279 792 448 958
1·235	·919 285 224	5 201	606	0·245 503 822	1·70	·062 711 040 496 868	4·0	·(5)0 126 982 346 719
36	9 530 425	4 595	5	4 897 931	1·72	·058 565 015 701 018	4·1	·(5)0 056 489 121 206
37	9 775 020	3 991	4	4 293 047	1·74	·054 649 360 707 757	4·2	·(5)0 024 632 040 987
38	·920 019 011	3 388	3	3 689 169	1·76	·050 954 726 185 724	4·3	·(5)0 010 528 102 122
39	0 262 399	242 785	2	3 086 298	1·78	·047 471 879 092 242	4·4	·(8)4 410 764 094 683
1·240	·920 505 184	2 184	601	0·242 484 434	1·80	·044 191 723 332 011	4·5	·(8)1 811 305 895 909
41	0 747 368	1 584	0	1 883 575	1·82	·041 105 318 483 320	4·6	·(8)0 729 094 500 238
42	0 988 952	0 984	599	1 283 723	1·84	·038 203 896 637 112	4·7	·(8)0 287 666 940 281
43	1 229 936	0 386	8	0 684 878	1·86	·035 478 877 401 325	4·8	·(8)0 111 252 606 898
44	1 470 322	239 789	7	0 087 038	1·88	·032 921 881 129 160	4·9	·(8)0 042 173 976 220
1·245	·921 710 110	9 192	596	0·239 490 205	1·90	·030 524 740 435 448	5·0	·(8)0 015 670 866 531
46	1 949 303	8 597	5	8 894 377	1·92	·028 279 510 069 901	5·1	·(11)5 707 627 016 929
47	2 187 900	8 003	4	8 299 555	1·94	·026 178 475 220 036	5·3	·(11)0 713 055 054 375
48	2 425 902	7 409	3	7 705 739	1·96	·024 214 158 319 685	5·5	·(11)0 082 233 160 452
49	2 663 311	236 817	592	7 112 928	1·98	·022 379 324 441 554	6·0	·(14)0 261 730 123 925

* The figures in parentheses indicate the number of ciphers between the decimal point and the figures that follow.

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t=1·000$ TO $t=3·000$.

$t=1·000$] [1·049

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 +	$\log \frac{2}{\sqrt{\pi}}e^{-t^2}+10.$
		415 522 743 218		829 383		
1·000	0·842 700 792 949 715		830 214 856		1660	9·618 160 577 413 6624
1	3 115 485 478 076	4 692 528 361	829 383 813	831 043	1652	7 291 554 155 3740
2	3 529 348 622 624	3 863 144 548	8 551 118	2 695	1644	6 421 662 308 1218
3	3 942 383 216 054	3 034 593 430	7 716 779	4 339	1636	5 550 901 871 9058
4	4 354 590 092 706	2 206 876 651	6 880 804	5 975	1627	4 679 272 846 7260
		411 379 995 847		837 602		
1·005	0·844 765 970 088 552		826 043 202		1619	9·613 806 775 232 5823
6	5 176 524 041 197	0 553 952 644	5 203 982	9 221	1611	2 933 409 029 4749
7	5 586 252 789 859	409 728 748 663	4 363 150	840 832	1602	2 059 174 237 4037
8	5 995 157 175 372	8 904 385 513	3 520 716	2 434	1594	1 184 070 856 3686
9	6 403 238 040 168	8 080 864 796	2 676 688	4 028	1586	0 308 098 886 3697
		407 [illegible]58 188 108		845 614		
1·010	0·846 810 496 228 277		821 831 074		1578	9·609 431 258 327 4071
11	7 216 932 585 311	6 436 357 034	0 983 883	7 192	1569	8 553 549 179 4806
12	7 622 547 958 462	5 615 273 152	0 135 122	8 761	1561	7 674 971 442 5903
13	8 027 343 196 492	4 795 238 030	819 284 800	850 322	1553	6 795 525 116 7362
14	8 431 319 149 722	3 975 953 230	8 432 925	1 875	1545	5 915 210 201 9184
		403 157 520 304		853 419		
1·015	0·848 834 476 670 026		817 579 506		1536	9·605 034 026 698 1367
16	9 236 816 610 825	2 339 940 798	6 724 550	4 956	1528	4 151 974 605 3911
17	9 638 339 827 073	1 523 216 248	5 868 067	6 484	1520	3 269 053 923 6818
18	0·850 039 047 175 254	0 707 348 181	5 010 063	8 003	1512	2 385 264 653 0087
19	0 438 939 513 372	399 892 338 118	4 150 548	9 515	1503	1 500 606 793 3718
		399 078 187 570		861 018		
1·020	0·850 838 017 700 942		813 289 530		1495	9·600 615 080 344 7711
21	1 236 282 598 982	8 264 898 040	[illegible] 427 016	2 514	1487	9·599 728 685 307 2065
22	1 633 735 070 006	7 452 471 024	1 563 016	4 001	1479	8 841 421 680 6782
23	2 030 375 978 015	6 640 908 008	[illegible] 697 536	5 479	1471	7 953 289 465 1860
24	2 426 206 188 487	5 830 210 472	809 830 586	6 950	1462	7 064 288 660 7301
		395 020 379 886		868 412		
1·025	0·852 821 226 568 372		808 962 174		1454	9·596 174 419 267 3103
26	3 215 437 986 084	4 211 417 711	8 092 308	9 866	1446	5 283 681 284 9267
27	3 608 841 311 487	3 403 325 403	7 220 996	871 312	1438	4 392 074 713 5794
28	4 001 437 415 894	2 596 104 407	6 348 246	2 750	1430	3 499 599 553 2682
29	4 393 227 172 054	1 789 756 161	5 474 067	4 179	1421	2 606 255 803 9932
		390 984 282 094		875 601		
1·030	0·854 784 211 454 148		804 598 466		1413	9·591 712 043 465 7544
31	5 174 391 137 776	0 179 683 628	3 721 452	7 014	1405	0 816 962 538 5518
32	5 563 767 099 952	389 375 962 176	[illegible] 843 033	8 419	1397	9·589 921 013 022 3854
33	5 952 340 219 095	8 573 119 143	[illegible] 963 217	9 816	1389	9 024 194 917 2552
34	6 340 111 375 020	7 771 155 925	1 082 013	881 205	1381	8 126 508 223 1612
		386 970 073 913		882 585		
1·035	0·856 727 081 448 933		800 199 427		1372	9·587 227 952 940 1033
36	7 113 251 323 418	6 169 874 485	799 315 470	3 958	1364	6 328 528 068 0817
37	7 498 621 882 434	5 370 559 015	8 430 148	5 322	1356	5 428 236 607 0963
38	7 883 194 011 301	4 572 128 867	7 543 470	6 678	1348	4 527 075 557 1470
39	8 266 968 596 698	3 774 585 397	6 655 444	8 026	1340	3 625 045 918 2340
		382 977 929 953		889 366		
1·040	0·858 649 946 526 651		795 766 078		1332	9·582 722 147 690 3571
41	9 032 128 690 527	2 182 163 875	4 875 380	890 698	1324	1 818 380 873 5164
42	9 413 515 979 022	1 387 288 495	3 983 359	2 021	1316	0 913 745 467 7120
43	9 794 109 284 158	0 593 305 136	3 090 022	3 337	1307	0 008 241 472 9437
44	0·860 173 909 499 272	379 800 215 114	2 195 377	4 645	1299	9·579 101 868 889 2116
		379 008 019 737		895 944		
1·045	0·860 552 917 519 009		791 299 433		1291	9·578 194 627 716 5157
46	0 931 134 239 312	8 216 720 303	0 402 198	7 235	1283	7 286 517 954 8560
47	1 308 560 557 417	7 426 318 105	789 503 680	8 518	1275	6 377 539 604 2325
48	1 685 197 371 843	6 636 814 426	8 603 886	9 794	1267	5 467 692 664 6452
49	2 061 045 582 382	5 848 210 540	7 702 825	901 061	1259	4 556 977 136 0941
		375 060 507 714		902 320		

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t=1·000$ TO $t=3·000$.

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 +	$\log \frac{2}{\sqrt{\pi}}e^{-t^2}+10$.
1·050	0·862 436 106 090 097	374 273 707 209	786 800 505	903 571	1251	9·573 645 393 018 5791
51	2 810 379 797 306	3 487 810 274	5 896 935	4 814	1243	2 732 940 312 1004
52	3 183 867 607 580	2 702 818 154	4 992 121	6 049	1235	1 819 619 016 6579
53	3 556 570 425 734	1 918 732 082	4 086 072	7 276	1227	0 905 429 132 2515
54	3 928 489 157 815	371 135 553 286	3 178 796	908 495	1219	9·569 990 370 658 8814
1·055	0·864 299 624 711 101	0 353 282 984	782 270 302	9 705	1211	9·569 074 443 596 5474
56	4 669 977 994 085	369 571 922 388	1 360 596	910 908	1203	8 157 647 945 2496
57	5 039 549 916 473	8 791 472 700	0 449 688	2 103	1195	7 239 983 704 9881
58	5 408 341 389 172	8 011 935 115	779 537 585	3 290	1187	6 321 450 875 7627
59	5 776 353 324 288	367 233 310 821	8 624 295	914 469	1179	5 402 049 457 5735
1·060	0·866 143 586 635 108	6 455 600 995	777 709 826	5 640	1171	9·564 481 779 450 4205
61	6 510 042 236 103	5 678 806 809	6 794 186	6 803	1163	3 560 640 854 3037
62	6 875 721 042 912	4 902 929 426	5 877 383	7 958	1155	2 638 633 669 2231
63	7 240 623 972 338	4 127 970 001	4 959 425	9 105	1147	1 715 757 895 1787
64	7 604 751 942 340	363 353 929 681	4 040 320	920 244	1139	0 792 013 532 1705
1·065	0·867 968 105 872 021	2 580 809 605	773 120 076	1 375	1131	9·559 867 400 580 1985
66	8 330 686 681 626	1 808 610 904	2 198 701	2 498	1123	8 941 919 039 2626
67	8 692 495 292 529	1 037 334 701	1 276 203	3 614	1115	8 015 568 909 3630
68	9 053 532 627 231	0 266 982 112	0 352 589	4 721	1107	7 088 350 190 4996
69	9 413 799 609 343	359 497 554 244	769 427 868	925 821	1099	6 160 262 882 6723
1·070	0·869 773 297 163 587	8 729 052 196	768 502 048	6 912	1092	9·555 231 306 985 8813
71	0·870 132 026 215 783	7 961 477 061	7 575 135	7 996	1084	4 301 482 500 1264
72	0 489 987 692 844	7 194 829 921	6 647 140	9 072	1076	3 370 789 425 4077
73	0 847 182 522 766	6 429 111 853	5 718 068	930 140	1068	2 439 227 761 7252
74	1 203 611 634 619	355 664 323 925	4 787 929	931 200	1060	1 506 797 509 0790
1·075	0·871 559 275 958 544	4 900 467 196	763 856 729	2 252	1052	9·550 573 498 667 4689
76	1 914 176 425 740	4 137 542 719	2 924 477	3 296	1044	9·549 639 331 236 8950
77	2 268 313 968 459	3 375 551 538	1 991 181	4 333	1037	8 704 295 217 3573
78	2 621 689 519 997	2 614 494 690	1 056 848	5 361	1029	7 768 390 608 8558
79	2 974 304 014 687	351 854 373 203	0 121 487	936 382	1021	6 831 617 411 3905
1·080	0·873 326 158 387 890	1 095 188 099	759 185 105	7 395	1013	9·545 893 975 624 9613
81	3 677 253 575 988	0 336 940 389	8 247 709	8 401	1005	4 955 465 249 5684
82	4 027 590 516 377	349 579 631 081	7 309 309	9 398	997	4 016 086 285 2117
83	4 377 170 147 458	8 823 261 170	6 369 911	940 388	990	3 075 838 731 8911
84	4 725 993 408 628	348 067 831 647	5 429 523	941 370	982	2 134 722 589 6068
1·085	0·875 074 061 240 276	7 313 343 494	754 488 153	2 344	974	9·541 192 737 858 3586
86	5 421 374 583 770	6 559 797 685	3 545 809	3 310	966	0 249 884 538 1467
87	5 767 934 381 455	5 807 195 185	2 602 499	4 269	959	9·539 306 162 628 9709
88	6 113 741 576 640	5 055 536 955	1 658 231	5 220	951	8 361 572 130 8313
89	6 458 797 113 595	344 304 823 943	0 713 011	946 163	943	7 416 113 043 7280
1·090	0·876 803 101 937 538	3 555 057 095	749 766 848	7 098	935	9·536 469 785 367 6608
91	7 146 656 994 633	2 806 237 345	8 819 750	8 026	928	5 522 589 102 6298
92	7 489 463 231 978	2 058 365 620	7 871 725	8 946	920	4 574 524 248 6350
93	7 831 521 597 598	1 311 442 841	6 922 779	9 858	912	3 625 590 805 6764
94	8 172 833 040 440	340 565 469 921	5 972 921	950 763	905	2 675 788 773 7540
1·095	0·878 513 398 510 361	339 820 447 763	745 022 158	1 660	897	9·531 725 118 152 8677
96	8 853 218 958 123	9 076 377 265	4 070 498	2 549	889	0 773 578 943 0177
97	9 192 295 335 388	8 333 259 315	3 117 949	3 431	882	9·529 821 171 144 2039
98	9 530 628 594 703	7 591 094 796	2 164 519	4 304	874	8 867 894 756 4262
99	9 868 219 689 500	336 849 884 582	1 210 214	955 171	866	7 913 749 779 6848

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2}dt$. (2) FROM $t = 1·000$ TO $t = 3·000$.

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 +	$\log \frac{2}{\sqrt{\pi}}\epsilon^{-t^2} + 10.$
1·100	0·880 205 069 574 082	336 109 629 538	740 255 044	956 030	859	9·526 958 736 213 9796
1	0 541 179 203 620	5 370 330 524	739 299 014	6 881	851	6 002 854 059 3105
2	0 876 549 534 144	4 631 988 391	8 342 134	7 724	843	5 046 103 315 6776
3	1 211 181 522 535	3 894 603 981	7 384 409	8 560	836	4 088 483 983 0810
4	1 545 076 126 517	333 158 178 132	6 425 850	959 388	828	3 129 996 061 5205
1·105	0·881 878 234 304 648	2 422 711 671	735 466 461	960 209	821	9·522 170 639 550 9962
6	2 210 657 016 319	1 688 205 418	4 506 252	1 022	813	1 210 414 451 5081
7	2 542 345 221 737	0 954 660 188	3 545 230	1 828	806	0 249 320 763 0562
8	2 873 299 881 925	0 222 076 785	2 583 403	2 626	798	9·519 287 358 485 6405
9	3 203 521 958 710	329 490 456 008	1 620 777	963 416	791	8 324 527 619 2610
1·110	0·883 533 012 414 718	8 759 798 647	730 657 361	4 199	783	9·517 360 828 163 9177
11	3 861 772 213 365	8 030 105 486	729 693 162	4 975	775	6 396 260 119 6106
12	4 189 802 318 851	7 301 377 299	8 728 187	5 743	768	5 430 823 486 3396
13	4 517 103 696 150	6 573 614 854	7 762 444	6 503	760	4 464 518 264 1049
14	4 843 677 311 004	325 846 818 913	6 795 941	967 256	753	3 497 344 452 9064
1·115	0 885 169 524 129 917	5 120 990 227	725 828 685	8 002	746	9·512 529 302 052 7440
16	5 494 645 120 144	4 396 129 543	4 860 684	8 740	738	1 560 391 063 6179
17	5 819 041 249 687	3 672 237 599	3 891 944	9 470	731	0 590 611 485 5279
18	6 142 713 487 286	2 949 315 125	2 922 474	970 193	723	9·509 619 963 318 4741
19	6 465 662 802 411	322 227 362 844	1 952 281	970 909	716	8 648 446 562 4566
1·120	0·886 787 890 165 255	1 506 381 472	720 981 372	1 617	708	9·507 676 061 217 4752
21	7 109 396 546 726	0 786 371 717	0 009 755	2 318	701	6 702 807 283 5300
22	7 430 182 918 443	0 067 334 280	719 037 437	3 012	693	5 728 684 760 2099
23	7 750 250 252 724	319 349 269 856	8 064 425	3 698	686	4 753 693 648 7482
24	8 069 599 522 579	318 632 179 128	7 090 727	974 376	679	3 777 833 947 9116
1·125	0·888 388 231 701 708	7 916 062 778	716 116 351	5 048	671	9·502 801 105 658 1112
26	8 706 147 764 485	7 200 921 475	5 141 303	5 712	664	1 823 508 779 3470
27	9 023 348 685 960	6 486 755 884	4 165 591	6 369	657	0 845 043 311 6189
28	9 339 835 441 844	5 773 566 661	3 189 222	7 018	649	9·499 865 709 254 9271
29	9 655 609 008 506	315 061 354 457	2 212 205	977 660	642	8 885 506 609 2715
1·130	0·889 970 670 362 962	4 350 119 912	711 234 545	8 295	635	9·497 904 435 374 6520
31	0·890 285 020 482 874	3 639 863 662	0 256 250	8 922	627	6 922 495 551 0688
32	0 598 660 346 536	2 930 586 334	709 277 328	9 542	620	5 939 687 138 5217
33	0 911 590 932 870	2 222 288 547	8 297 786	980 155	613	4 956 010 137 0108
34	1 223 813 221 417	311 514 970 916	7 317 631	980 761	606	3 971 464 546 5362
1·135	0·891 535 328 192 333	0 808 634 046	706 336 871	1 359	598	9·492 986 050 367 0977
36	1 846 136 826 379	0 103 278 534	5 355 512	1 950	591	1 999 767 598 6954
37	2 156 240 104 913	309 398 904 972	4 373 562	2 534	584	1 012 616 241 3293
38	2 465 639 009 885	8 695 513 944	3 391 028	3 111	577	0 024 596 294 9994
39	2 774 334 523 830	307 993 106 027	2 407 917	983 680	569	9·489 035 707 759 7057
1·140	0·893 082 327 629 857	7 291 681 790	701 424 237	4 242	562	9·488 045 950 635 4482
41	3 389 619 311 646	6 591 241 795	0 439 995	4 797	555	7 055 324 922 2269
42	3 696 210 553 441	5 891 786 597	699 455 198	5 345	548	6 063 830 620 0418
43	4 002 102 340 038	5 193 316 744	8 469 853	5 886	541	5 071 467 728 8928
44	4 307 295 656 782	304 495 832 778	7 483 967	986 420	534	4 078 236 248 7801
1·145	0·894 611 791 489 560	3 799 335 231	696 497 547	6 946	526	9·483 084 136 179 7035
46	4 915 590 824 791	3 103 824 630	5 510 601	7 465	519	2 089 167 521 6632
47	5 218 694 649 421	2 409 301 495	4 523 135	7 978	512	1 093 330 274 6590
48	5 521 103 950 916	1 715 766 337	3 535 158	8 483	505	0 096 624 438 6911
49	5 822 819 717 253	301 023 219 662	2 546 675	988 981	498	9·479 099 050 013 7593

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt.$ (2) FROM $t = 1{\cdot}000$ TO $t = 3{\cdot}000$.

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 +	$\log \frac{2}{\sqrt{\pi}}e^{-t^2} + 10.$
1·150	0·896 123 842 936 915	300 331 661 969	691 557 694	989 472	491	9·478 100 606 999 8637
51	6 424 174 598 884	299 641 093 747	0 568 222	9 956	484	7 101 295 397 0043
52	6 723 815 692 630	8 951 515 481	689 578 266	990 433	477	6 101 115 205 1812
53	7 022 767 208 111	8 262 927 648	8 587 833	0 903	470	5 100 066 424 3942
54	7 321 030 135 759	297 575 330 718	7 596 930	991 366	463	4 098 049 054 6434
1·155	0·897 618 605 466 477	6 888 725 153	686 605 565	1 821	456	9·473 095 363 095 9287
56	7 915 494 191 630	6 203 111 409	5 613 743	2 270	449	2 091 708 548 2503
57	8 211 697 303 039	5 518 489 936	4 621 473	2 712	442	1 087 185 411 6081
58	8 507 215 792 976	4 834 861 175	3 628 761	3 147	435	0 081 793 686 0021
59	8 802 050 654 151	294 152 225 561	2 635 614	993 575	428	9·469 075 533 371 4322
1·160	0·899 096 202 879 712	3 470 583 522	681 642 039	3 996	421	9·468 068 404 467 8986
61	9 389 673 463 234	2 789 935 479	0 648 043	4 410	414	7 060 406 975 4012
62	9 682 463 398 713	2 110 281 845	679 653 633	4 817	407	6 051 540 893 9399
63	9 974 573 680 558	1 431 623 029	8 658 816	5 217	400	5 041 806 223 5148
64	0·900 266 005 303 587	290 753 959 430	7 663 599	995 611	393	4 031 202 964 1260
1·165	0·900 556 759 263 017	0 077 291 442	676 667 988	5 997	386	9·463 019 731 115 7733
66	0 846 836 554 459	289 401 619 450	5 671 991	6 377	380	2 007 390 678 4568
67	1 136 238 173 909	8 726 943 836	4 675 615	6 749	373	0 994 181 652 1765
68	1 424 965 117 745	8 053 264 970	3 678 865	7 115	366	9·459 980 104 036 9325
69	1 713 018 382 715	287 380 583 220	2 681 750	997 474	359	8 965 157 832 7245
1·170	0·902 000 398 965 936	6 708 898 945	671 684 276	7 827	352	9·457 949 343 039 5528
71	2 287 107 864 880	6 038 212 496	0 686 449	8 172	345	6 932 659 657 4173
72	2 573 146 077 376	5 368 524 218	669 688 277	8 511	339	5 915 107 686 3180
73	2 858 514 601 595	4 699 834 452	8 689 767	8 842	332	4 896 687 126 2549
74	3 143 214 436 046	284 032 143 528	7 690 924	999 168	325	3 877 397 977 2280
1·175	0·903 427 246 579 574	3 365 451 771	666 691 757	9 486	318	9·452 857 240 239 2372
76	3 710 612 031 345	2 699 759 501	5 692 271	9 798	312	1 836 213 912 2827
77	3 993 311 790 846	2 035 067 027	4 692 473	1000 102	305	0 814 318 996 3643
78	4 275 346 857 873	1 371 374 657	3 692 371	0 401	298	9·449 791 555 491 4822
79	4 556 718 232 530	280 708 682 687	2 691 970	1000 692	291	8 767 923 397 6362
1·180	0·904 837 426 915 217	0 046 991 409	661 691 278	0 977	285	9·447 743 422 714 8264
81	5 117 473 906 625	279 386 301 107	0 690 301	1 255	278	6 718 053 443 0528
82	5 396 860 207 733	8 726 612 061	659 689 046	1 526	271	5 691 815 582 3155
83	5 675 586 819 794	8 067 924 542	8 687 520	1 791	265	4 664 709 132 6143
84	5 953 654 744 336	277 410 238 813	7 685 728	1002 049	258	3 636 734 093 9493
1·185	0·906 231 064 983 149	6 753 555 134	656 683 679	2 301	252	9·442 607 890 466 3205
86	6 507 818 538 283	6 097 873 756	5 681 378	2 546	245	1 578 178 249 7279
87	6 783 916 412 039	5 443 194 924	4 678 832	2 784	238	0 547 597 444 1714
88	7 059 359 606 963	4 789 518 877	3 676 048	3 016	232	9·439 516 148 049 6512
89	7 334 149 125 840	274 136 845 845	2 673 031	1003 241	225	8 483 830 066 1672
1·190	0·907 608 285 971 685	3 485 176 055	651 669 790	3 460	219	9·437 450 643 493 7194
91	7 881 771 147 741	2 834 509 726	0 666 330	3 672	212	6 416 588 332 3077
92	8 154 605 657 466	2 184 847 068	649 662 657	3 878	206	5 381 664 581 9323
93	8 426 790 504 535	1 536 188 289	8 658 779	4 077	199	4 345 872 242 5930
94	8 698 326 692 824	270 888 533 587	7 654 702	1004 270	193	3 309 211 314 2899
1·195	0·908 969 215 226 411	0 241 883 155	646 650 432	4 456	186	9·432 271 681 797 0231
96	9 239 457 109 565	269 596 237 178	5 645 976	4 636	180	1 233 283 690 7924
97	9 509 053 346 743	8 951 595 837	4 641 341	4 809	173	0 194 016 995 5979
98	9 778 004 942 581	8 307 959 305	3 636 532	4 976	167	9·429 153 881 711 4396
99	0·910 046 312 901 886	267 665 327 749	2 631 556	1005 136	160	8 112 877 838 3175

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t=1\cdot000$ TO $t=3\cdot000$.

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 + −	$\log \frac{2}{\sqrt{\pi}}e^{-t^2}+10.$
1·200	0·910 313 978 229 635	267 023 701 329	641 626 420	1005 290	+154	9·427 071 005 376 2316
1	0 581 001 930 965	6 383 080 199	0 621 130	438	148	6 028 264 325 1819
2	0 847 385 011 164	5 743 464 507	639 615 692	579	141	4 984 654 685 1684
3	1 113 128 475 671	5 104 854 394	8 610 113	714	135	3 940 176 456 1911
4	1 378 233 330 066	264 467 249 995	7 604 399	1005 842	129	2 894 829 638 2500
1·205	0·911 642 700 580 061	3 830 651 438	636 598 557	964	122	9·421 848 614 231 3450
6	1 906 531 231 498	3 195 058 845	5 592 593	1006 080	116	0 801 530 235 4763
7	2 169 726 290 344	2 560 472 333	4 586 512	190	110	9·419 753 577 650 6437
8	2 432 286 762 677	1 926 892 011	3 580 322	293	103	8 704 756 476 8474
9	2 694 213 654 688	261 294 317 982	2 574 029	1006 390	97	7 655 066 714 0872
1·210	0·912 955 507 972 669	0 662 750 342	631 567 639	481	91	9·416 604 508 362 3633
11	0·913 216 170 723 012	0 032 189 184	0 561 158	565	84	5 553 081 421 6755
12	476 202 912 196	259 402 634 591	629 554 593	643	78	4 500 785 892 0239
13	735 605 546 786	8 774 086 641	8 547 950	715	72	3 447 621 773 4085
14	994 379 633 427	258 146 545 406	7 541 234	1006 781	66	2 393 589 065 8293
1·215	0·914 252 526 178 834	7 520 010 953	626 534 453	841	60	9·411 338 687 769 2863
16	510 046 189 787	6 894 483 341	5 527 612	894	53	0 282 917 883 7795
17	766 940 673 128	6 269 962 622	4 520 718	941	47	9·409 226 279 409 3089
18	0·915 023 210 635 750	5 646 448 845	3 513 777	983	41	8 168 772 345 8745
19	278 857 084 596	255 023 942 051	2 506 794	1007 017	35	7 110 396 693 4763
1·220	0·915 533 881 026 647	4 402 442 274	621 499 777	046	29	9·406 051 152 452 1142
21	788 283 468 921	3 781 949 543	0 492 731	069	23	4 991 039 621 7884
22	0·916 042 065 418 464	3 162 463 882	619 485 662	086	17	3 930 058 202 4988
23	295 227 882 346	2 543 985 306	8 478 576	096	11	2 868 208 194 2453
24	547 771 867 652	251 926 513 826	7 471 480	1007 101	+5	1 805 489 597 0281
1·225	0·916 799 698 381 479	1 310 049 447	616 464 379	099	−2	9·400 741 902 410 8470
26	0·917 051 008 430 926	0 694 592 168	5 457 280	092	8	9·399 677 446 635 7021
27	301 703 023 094	0 080 141 980	4 450 188	078	14	8 612 122 271 5934
28	551 783 165 073	249 466 698 870	3 443 110	058	20	7 545 929 318 5210
29	801 249 863 943	248 854 262 818	2 436 051	1007 033	26	6 478 867 776 4847
1·230	0·918 050 104 126 761	8 242 833 800	611 429 019	001	32	9·395 410 937 645 4846
31	298 346 960 561	7 632 411 782	0 422 017	1006 964	38	4 342 138 925 5207
32	545 979 372 343	7 022 996 729	609 415 053	920	43	3 272 471 616 5930
33	793 002 369 072	6 414 588 596	8 408 133	871	49	2 201 935 718 7015
34	0·919 039 416 957 668	245 807 187 334	7 401 262	1006 816	55	1 130 531 231 8461
1·235	0·919 285 224 145 002	5 200 792 888	606 394 446	755	61	9·390 058 258 156 0270
36	530 424 937 890	4 595 405 196	5 387 692	687	67	9·388 985 116 491 2441
37	775 020 343 087	3 991 024 192	4 381 004	615	73	7 911 106 237 4973
38	0·920 019 011 367 279	3 387 649 802	3 374 390	536	79	6 836 227 394 7868
39	262 399 017 081	242 785 281 949	2 367 854	1006 451	85	5 760 479 963 1124
1·240	0·920 505 184 299 030	2 183 920 546	601 361 403	361	90	9·384 683 863 942 4743
41	747 368 219 576	1 583 565 504	0 355 042	264	96	3 606 379 332 8723
42	988 951 785 080	0 984 216 727	599 348 777	162	102	2 528 026 134 3065
43	0·921 229 936 001 806	0 385 874 112	8 342 615	055	108	1 448 804 346 7769
44	470 321 875 918	239 788 537 551	7 336 560	1005 941	114	0 368 713 970 2835
1·245	0·921 710 110 413 469	9 192 206 932	596 330 619	822	119	9·379 287 755 004 8264
46	949 302 620 401	8 596 882 134	5 324 798	697	125	8 205 927 450 4054
47	0·922 187 899 502 535	8 002 563 033	4 319 101	566	131	7 123 231 307 0205
48	425 902 065 568	7 409 249 498	3 313 535	429	136	6 039 666 574 6719
49	663 311 315 066	236 816 941 392	2 308 106	1005 287	142	4 955 233 253 3595

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t=1·000$ TO $t=3·000$.

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	$\log.\frac{2}{\sqrt{\pi}}e^{-t^2}+10.$
1·250	0·922 900 128 256 458	236 225 638 574	591 302 818	1005 139	148	9·373 869 931 343 0829
51	0·923 136 353 895 032	5 635 340 895	0 297 679	1004 986	153	2 783 760 843 8433
52	371 989 235 927	5 046 048 202	589 292 693	827	159	1 696 721 755 6394
53	607 035 284 129	4 457 760 336	8 287 866	662	165	0 608 814 078 4718
54	841 493 044 464	233 870 477 132	7 283 204	1004 492	170	9·369 520 037 812 3403
1·255	0·924 075 363 521 596	3 284 198 419	586 278 712	316	176	9·368 430 392 957 2451
56	308 647 720 015	2 698 924 023	5 274 396	134	182	7 339 879 513 1860
57	541 346 644 038	2 114 653 761	4 270 262	1003 947	187	6 248 497 480 1631
58	773 461 297 799	1 531 387 446	3 266 315	755	193	5 156 246 858 1764
59	0·925 004 992 685 244	230 949 124 885	2 262 560	1003 556	198	4 063 127 647 2260
1·260	0·925 235 941 810 130	0 367 865 881	581 259 004	353	204	9·362 969 139 847 3117
61	466 309 676 011	229 787 610 230	0 255 651	144	209	1 874 283 458 4336
62	696 097 286 241	9 208 357 723	579 252 508	1002 929	215	0 778 558 480 5917
63	925 305 643 964	8 630 108 144	8 249 579	709	220	9·359 681 964 913 7860
64	0·926 153 935 752 108	228 052 861 275	7 246 870	1002 483	226	8 584 502 758 0164
1·265	0·926 381 988 613 383	7 476 616 888	576 244 386	252	231	9·357 486 172 013 2831
66	609 465 230 271	6 901 374 754	5 242 134	016	236	6 386 972 679 5860
67	836 366 605 025	6 327 134 635	4 240 118	1001 774	242	5 286 904 756 9250
68	0·927 062 693 739 660	5 753 896 291	3 238 344	527	247	4 185 968 245 3003
69	288 447 635 951	225 181 659 473	2 236 818	1001 274	253	3 084 163 144 7118
1·270	0·927 513 629 295 425	4 610 423 930	571 235 544	016	258	9·351 981 489 455 1594
71	738 239 719 354	4 040 189 402	0 234 527	1000 753	263	0 877 947 176 6432
72	962 279 908 757	3 470 955 628	569 233 775	484	269	9·349 773 536 309 1633
73	0·928 185 750 864 384	2 902 722 337	8 233 290	210	274	8 668 256 852 7195
74	408 653 586 721	222 335 489 257	7 233 080	999 931	279	7 562 108 807 3119
1·275	0·928 630 989 075 979	1 769 256 108	566 233 149	9 646	285	9·346 455 092 172 9405
76	852 758 332 086	1 204 022 605	5 233 503	9 357	290	5 347 206 949 6053
77	0·929 073 962 354 692	0 639 788 459	4 234 146	9 062	295	4 238 453 137 3063
78	294 602 143 151	0 076 553 375	3 235 084	8 761	300	3 128 830 736 0435
79	514 678 696 526	219 514 317 052	2 236 323	998 456	305	2 018 339 745 8169
1·280	0·929 734 193 013 578	8 953 079 185	561 237 867	8 145	311	9·340 906 980 166 6265
81	953 146 092 763	8 392 839 463	0 239 722	7 829	316	9·339 794 751 998 4722
82	0·930 171 538 932 226	7 833 597 570	559 241 893	7 508	321	8 681 655 241 3542
83	389 372 529 796	7 275 353 186	8 244 384	7 182	326	7 567 689 895 2724
84	606 647 882 982	216 718 105 983	7 247 202	996 851	331	6 452 855 960 2267
1·285	0·930 823 365 988 965	6 161 855 631	556 250 352	6 514	336	9·335 337 153 436 2173
86	0·931 039 527 344 596	5 606 601 794	5 253 838	6 173	342	4 220 582 323 2440
87	255 134 446 390	5 052 344 129	4 257 665	5 826	347	3 103 142 621 3069
88	470 186 790 519	4 499 082 290	3 261 839	5 474	352	1 984 834 330 4061
89	684 685 872 809	213 946 815 925	2 266 365	995 117	357	0 865 657 450 5414
1·290	0·931 898 632 688 734	3 395 544 677	551 271 248	4 756	362	9·329 745 611 981 7129
91	0·932 112 028 233 411	2 845 268 185	0 276 492	4 389	367	8 624 697 923 9206
92	324 873 501 596	2 295 986 082	549 282 103	4 017	372	7 502 915 277 1645
93	537 169 487 678	1 747 697 995	8 288 087	3 640	377	6 380 264 041 4446
94	748 917 185 674	211 200 403 549	7 294 447	993 258	382	5 256 744 216 7609
1·295	0·932 960 117 589 223	0 654 102 360	546 301 189	2 871	387	9·324 132 355 803 1134
96	0·933 170 771 691 583	0 108 794 042	5 308 318	2 479	392	3 007 098 800 5020
97	380 880 485 625	209 564 478 204	4 315 838	2 083	397	1 880 973 208 9269
98	590 444 963 829	9 021 154 449	3 323 755	1 681	402	0 753 979 028 3880
99	799 466 118 278	208 478 822 374	2 332 074	991 275	407	9·319 626 116 258 8852

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t=1{\cdot}000$ TO $t=3{\cdot}000$.

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	$\log. \frac{2}{\sqrt{\pi}}e^{-t^2}+10.$
1·300	0·934 007 944 940 652	207 937 481 574	541 340 800	990 863	411	9·318 497 384 900 4187
1	215 882 422 227	7 397 131 638	0 349 937	0 447	416	7 367 784 952 9883
2	423 279 553 864	6 857 772 147	539 359 490	0 026	421	6 237 316 416 5942
3	630 137 326 012	6 319 402 683	8 369 464	989 600	426	5 105 979 291 2362
4	836 456 728 695	205 782 022 818	7 379 865	989 169	431	3 973 773 576 9144
1·305	0·935 042 238 751 513	5 245 632 123	536 390 696	8 733	436	9·312 840 699 273 6288
6	247 484 383 636	4 710 230 160	5 401 963	8 293	440	1 706 756 381 3794
7	452 194 613 796	4 175 816 490	4 413 670	7 848	445	0 571 944 900 1662
8	656 370 430 256	3 642 390 669	3 425 822	7 398	450	9·309 436 264 829 9892
9	860 012 820 955	203 109 952 245	2 438 424	986 943	455	8 299 716 170 8484
1·310	0·936 063 122 773 200	2 578 500 764	531 451 481	6 484	459	9·307 162 298 922 7438
11	265 701 273 963	2 048 035 767	0 464 997	6 020	464	6 024 013 085 6754
12	467 749 309 730	1 518 556 789	529 478 977	5 551	469	4 884 858 659 6432
13	669 267 866 519	0 990 063 363	8 493 426	5 077	473	3 744 835 644 6471
14	870 257 929 882	200 462 555 014	7 508 349	984 599	478	2 603 944 040 6873
1·315	0·937 070 720 484 896	199 936 031 264	526 523 750	4 117	483	9·301 462 183 847 7636
16	270 656 516 160	9 410 491 631	5 539 633	3 629	487	0 319 555 065 8762
17	470 067 007 791	8 885 935 627	4 556 004	3 137	492	9·299 176 057 695 0249
18	668 952 943 418	8 362 362 760	3 572 867	2 641	497	8 031 691 735 2098
19	867 315 306 178	197 839 772 534	2 590 226	982 139	501	6 886 457 186 4310
1·320	0·938 065 155 078 711	7 318 164 447	521 608 087	1 634	506	9·295 740 354 048 6883
21	262 473 243 158	6 797 537 994	0 626 453	1 123	510	4 593 382 321 9818
22	459 270 781 152	6 277 892 664	519 645 330	0 608	515	3 445 542 006 3115
23	655 548 673 815	5 759 227 942	8 664 722	0 089	519	2 296 833 101 6774
24	851 307 901 757	195 241 543 310	7 684 632	979 565	524	1 147 255 608 0795
1·325	0·939 046 549 445 067	4 724 838 243	516 705 067	9 037	528	9·289 996 809 525 5178
26	241 274 283 310	4 209 112 213	5 726 030	8 504	533	8 845 494 853 9923
27	435 483 395 522	3 694 364 687	4 747 526	7 967	537	7 693 311 593 5029
28	629 177 760 209	3 180 595 128	3 769 559	7 425	542	6 540 259 744 0498
29	822 358 355 336	192 667 802 994	2 792 134	976 879	546	5 386 339 305 6329
1·330	0·940 015 026 158 330	2 155 987 739	511 815 255	6 329	551	9·284 231 550 278 2521
31	207 182 146 070	1 645 148 813	0 838 926	5 774	555	3 075 892 661 9076
32	398 827 294 883	1 135 285 661	509 863 152	5 214	559	1 919 366 456 5992
33	589 962 580 544	0 626 397 723	8 887 938	4 651	564	0 761 971 662 3270
34	780 588 978 267	190 118 484 437	7 913 287	974 083	568	9·279 603 708 279 0911
1·335	0·940 970 707 462 704	189 611 545 233	506 939 204	3 511	572	9·278 444 576 306 8913
36	0·941 160 319 007 937	9 105 579 539	5 965 693	2 934	577	7 284 575 745 7277
37	349 424 587 476	8 600 586 780	4 992 759	2 353	581	6 123 706 595 6003
38	538 025 174 256	8 096 566 374	4 020 406	1 768	585	4 961 968 856 5091
39	726 121 740 630	187 593 517 736	3 048 638	971 179	589	3 799 362 528 4541
1·340	0·941 913 715 258 365	7 091 440 276	502 077 460	0 585	594	9·272 635 887 611 4353
41	0·942 100 806 698 641	6 590 333 402	1 106 875	969 987	598	1 471 544 105 4527
42	287 397 032 043	6 090 196 514	0 136 988	9 385	602	0 306 332 010 5062
43	473 487 228 557	5 591 029 011	499 167 503	8 779	606	9·269 140 251 326 5960
44	659 078 257 568	185 092 830 288	8 198 724	968 168	610	7 973 302 053 7220
1·345	0·942 844 171 087 856	4 595 599 732	497 230 556	7 554	615	9·266 805 484 191 8841
46	0·943 028 766 687 588	4 099 336 730	6 263 002	6 935	619	5 636 797 741 0825
47	212 866 024 318	3 604 040 663	5 296 067	6 312	623	4 467 242 701 3170
48	396 470 064 980	3 109 710 907	4 329 755	5 685	627	3 296 819 072 5878
49	579 579 775 887	182 616 346 837	3 364 070	965 054	631	2 125 526 854 8947

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t = 1\cdot000$ TO $t = 3\cdot000$.

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	log. $\frac{2}{\sqrt{\pi}}e^{-t^2} + 10$.
1·350	0·943 762 196 122 724	182 123 947 820	492 399 017	964 419	635	9·260 953 366 048 2378
51	944 320 070 544	1 632 513 222	1 434 598	3 779	639	9·259 780 336 652 6171
52	0·944 125 952 583 767	1 142 042 404	0 470 819	3 136	643	8 606 438 668 0326
53	307 094 626 170	0 652 534 721	489 507 683	2 489	647	7 431 672 094 4843
54	487 747 160 891	180 163 989 527	8 545 194	961 837	651	6 256 036 931 9722
1·355	0·944 667 911 150 418	179 676 406 171	487 583 357	1 182	655	9·255 079 533 180 4963
56	847 587 556 589	9 189 783 996	6 622 175	0 523	659	3 902 160 840 0566
57	·945 026 777 340 585	8 704 122 344	5 661 652	959 859	663	2 723 919 910 6531
58	205 481 462 930	8 219 420 552	4 701 793	9 192	667	1 544 810 392 2858
59	383 700 883 482	177 735 677 951	3 742 600	958 521	671	0 364 832 284 9546
1·360	0·945 561 436 561 433	7 252 893 872	482 784 080	7 846	675	9·249 183 985 588 6597
61	738 689 455 305	6 771 067 638	1 826 234	7 167	679	8 002 270 303 4009
62	915 460 522 943	6 290 198 571	0 869 067	6 484	683	6 819 686 429 1784
63	0·946 091 750 721 514	5 810 285 988	479 912 583	5 797	687	5 636 233 965 9920
64	267 561 007 502	175 331 329 201	8 956 786	955 106	691	4 451 912 913 8419
1·365	0·946 442 892 336 703	4 853 327 522	478 001 680	4 412	694	9·243 266 723 272 7279
66	617 745 664 224	4 376 280 254	7 047 268	3 714	698	2 080 665 042 6501
67	792 121 944 478	3 900 186 699	6 093 554	3 011	702	0 893 738 223 6085
68	966 022 131 177	3 425 046 156	5 140 543	2 306	706	9·239 705 942 815 6031
69	0·947 139 447 177 333	172 950 857 919	474 188 237	951 596	710	8 517 278 818 6339
1·370	0·947 312 398 035 252	2 477 621 277	473 236 642	0 882	713	9·237 327 746 232 7009
71	484 875 656 529	2 005 335 518	2 285 759	0 165	717	6 137 345 057 8041
72	656 880 992 047	1 533 999 924	1 335 594	949 444	721	4 946 075 293 9435
73	828 414 991 971	1 063 613 774	0 386 150	8 720	725	3 753 936 941 1191
74	999 478 605 746	170 594 176 345	469 437 430	947 991	728	2 560 929 999 3308
1·375	0·948 170 072 782 090	0 125 686 906	468 489 439	7 259	732	9·231 367 054 468 5788
76	340 198 468 996	169 658 144 727	7 542 179	6 524	736	0 172 310 348 8629
77	509 856 613 723	9 191 549 071	6 595 656	5 784	739	9·228 976 697 640 1833
78	679 048 162 794	8 725 899 199	5 649 871	5 041	743	7 780 216 342 5398
79	847 774 061 993	168 261 194 369	4 704 830	944 295	747	6 582 866 455 9326
1·380	0·949 016 035 256 363	7 797 433 834	463 760 535	3 545	750	9·225 384 647 980 3615
81	183 832 690 197	7 334 616 844	2 816 991	2 791	754	4 185 560 915 8266
82	351 167 307 040	6 872 742 644	1 874 200	2 034	757	2 985 605 262 3279
83	518 040 049 684	6 411 810 478	0 932 166	1 273	761	1 784 781 019 8654
84	684 451 860 162	165 951 819 584	459 990 894	940 508	764	0 583 088 188 4391
1·385	0·949 850 403 679 746	5 492 769 198	459 050 385	939 740	768	9·219 380 526 768 0490
86	0·950 015 896 448 944	5 034 658 553	8 110 645	8 969	771	8 177 096 758 6951
87	180 931 107 498	4 577 486 877	7 171 676	8 194	775	6 972 798 160 3774
88	345 508 594 375	4 121 253 394	6 233 482	7 415	778	5 767 630 973 0959
89	509 629 847 769	163 665 957 327	5 296 067	936 634	782	4 561 595 196 8505
1·390	0·950 673 295 805 096	3 211 597 894	454 359 433	5 848	785	9·213 354 690 831 6414
91	836 507 402 991	2 758 174 309	3 423 585	5 060	789	2 146 917 877 4685
92	999 265 577 299	2 305 685 783	2 488 526	4 267	792	0 938 276 334 3317
93	0·951 161 571 263 083	1 854 131 525	1 554 258	3 472	796	9·209 728 766 202 2312
94	323 425 394 608	161 403 510 739	0 620 786	932 673	799	8 518 387 481 1668
1·395	0·951 484 828 905 347	0 953 822 625	449 688 113	1 871	802	9·207 307 140 171 1386
96	645 782 727 972	0 505 066 383	8 756 243	1 065	806	6 095 024 272 1467
97	806 287 794 355	0 057 241 205	7 825 178	0 256	809	4 882 039 784 1909
98	966 345 035 560	159 610 346 283	6 894 922	929 444	812	3 668 186 707 2713
99	0·952 125 955 381 843	159 164 380 805	5 965 478	928 628	816	2 453 465 041 3879

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t=1·000$ TO $t=3·000$.

1·400] [1·449

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	log. $\frac{2}{\sqrt{\pi}}e^{-t^2}+10$.
1·400	0·952 285 119 762 649	158 719 343 956	445 036 850	927 800	819	9·201 237 874 786 5407
1	443 839 106 604	8 275 234 915	4 109 040	6 987	822	0 021 415 948 7897
2	602 114 341 580	7 832 051 862	3 182 053	6 162	825	9·198 804 088 509 0349
3	759 946 394 382	7 389 796 971	2 255 891	5 333	829	7 585 802 488 2162
4	917 336 191 352	156 948 466 412	1 330 558	924 501	832	6 366 827 877 5138
1·405	0·953 074 284 657 765	6 508 060 355	440 406 067	3 666	835	9·195 146 804 077 8476
6	230 792 718 120	6 068 577 965	439 482 391	2 828	838	3 926 092 889 2175
7	386 861 296 085	5 630 018 402	8 559 563	1 987	841	2 704 422 511 6237
8	542 491 314 487	5 192 380 826	7 637 576	1 142	845	1 481 883 345 0060
9	697 683 695 313	154 755 664 392	6 716 434	920 294	848	0 258 475 989 5446
1·410	0·953 852 439 359 705	4 319 868 253	435 796 140	919 443	851	9·189 034 109 845 0593
11	0·954 006 759 227 958	3 884 991 557	4 876 696	8 590	854	7 809 255 111 6102
12	160 644 219 515	3 451 033 450	3 958 107	7 732	857	6 583 941 789 1074
13	314 095 252 965	3 017 993 076	3 040 374	6 872	860	5 356 159 877 8207
14	467 113 246 041	152 585 860 574	2 123 502	916 009	863	4 128 409 377 4802
1·415	0·954 619 699 115 615	2 154 662 081	431 207 493	5 143	866	9·182 899 700 288 1759
16	771 853 777 696	1 724 369 732	0 292 350	4 274	869	1 670 302 809 0078
17	923 578 147 428	1 294 991 656	429 378 076	3 401	872	0 439 940 342 0759
18	0·955 074 873 139 084	0 866 520 981	8 464 675	2 526	875	9·179 208 721 486 4801
19	225 739 660 064	150 438 974 832	7 552 149	911 048	878	7 976 628 041 3206
1·420	0·955 376 178 640 896	0 012 334 331	426 640 501	0 766	881	9·176 743 666 007 1973
21	526 190 975 227	149 586 604 596	5 729 735	909 882	884	5 509 835 384 1101
22	675 777 579 823	9 161 784 744	4 819 852	8 995	887	4 275 130 172 0502
23	824 939 364 567	8 737 873 886	3 910 857	8 105	890	3 039 508 371 0444
24	973 677 238 453	148 314 871 134	3 002 752	907 212	893	1 803 131 981 0659
1·425	0·956 121 992 109 586	7 892 775 593	422 095 541	6 316	896	9·170 565 827 003 1235
26	269 884 885 179	7 471 586 369	1 189 224	5 417	899	9·169 327 053 434 2174
27	417 356 471 548	7 051 302 561	0 283 807	4 516	902	8 088 011 277 3474
28	564 407 774 109	6 631 923 270	419 379 292	3 611	904	6 848 700 531 5130
29	711 039 697 379	146 213 447 590	8 475 680	902 704	907	5 607 921 196 7100
1·430	0·956 857 253 144 969	5 795 874 013	417 572 977	1 794	910	9·164 366 273 272 9546
31	0·957 003 049 019 582	5 379 203 430	6 671 183	0 881	913	3 123 750 700 2894
32	148 428 223 012	4 963 433 128	5 770 302	899 965	916	1 880 371 058 5404
33	293 391 656 140	4 548 562 791	4 870 337	9 047	919	0 636 117 907 8876
34	437 940 218 931	144 134 591 501	3 971 291	898 125	921	9·159 390 995 688 2710
1·435	0·957 582 074 810 432	3 721 518 335	413 073 165	7 201	924	9·158 145 004 810 0005
36	725 796 328 767	3 309 342 371	2 175 964	6 275	927	6 898 145 365 1403
37	869 105 671 139	2 898 062 082	1 279 089	5 345	930	5 650 417 315 0382
38	0·958 012 003 733 221	2 487 678 338	0 384 344	4 413	932	4 401 820 680 1064
39	154 491 412 158	142 078 188 406	409 489 931	893 478	935	3 152 355 455 7307
1·440	0·958 296 569 600 565	1 669 501 953	408 596 453	2 541	937	9·151 902 021 042 3313
41	438 239 102 518	1 261 888 041	7 703 912	1 601	940	0 650 819 230 0680
42	579 501 080 559	0 855 075 789	6 812 311	0 658	943	9·149 398 748 248 0409
43	720 356 156 281	0 449 154 074	5 921 654	889 712	945	8 145 808 008 5501
44	860 805 310 361	140 044 122 133	5 031 942	888 764	948	6 892 000 490 0954
1·445	0·959 000 849 432 494	139 639 978 955	404 143 177	7 814	951	9·145 637 383 740 8760
46	140 489 411 449	9 236 723 592	3 255 364	6 801	953	4 381 778 393 0040
47	279 720 135 041	8 834 355 089	2 368 503	5 905	956	3 125 304 457 5485
48	418 560 490 129	8 432 872 491	1 482 595	4 047	958	1 868 081 938 4385
49	556 993 363 620	138 031 274 830	0 597 651	883 086	961	0 609 030 818 3048

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t = 1\cdot000$ TO $t = 3\cdot000$.

1·450] [1·499

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	$\log \frac{2}{\sqrt{\pi}}e^{-t^2} + 10$
1·450	0·959 [illegible] 637 459	137 632 561 174	399 713 665	[illegible]	963	9·139 350 911 125 3273
51	832 [illegible] 198 623	7 233 730 531	8 830 643	2 057	966	8 091 022 823 3260
52	969 891 929 164	6 835 781 945	7 948 586	1 088	968	6 830 265 942 3608
53	0·960 106 727 711 109	6 438 714 447	7 067 498	0 118	971	5 568 640 472 4319
54	243 166 425 556	136 042 527 067	6 187 380	879 144	973	4 306 146 413 5391
1·455	0·960 379 [illegible] 952 623	5 647 218 832	395 308 235	8 169	976	9·133 042 783 765 6826
56	[illegible]	5 252 188 765	4 [illegible]	7 191	[illegible]	[illegible]
57	[illegible]	4 [illegible] 889	3 [illegible]	6 215	[illegible]	[illegible]
58	724 [illegible]	4 466 559 224	2 [illegible]	5 227	[illegible]	9·129 247 484 [illegible]
59	929 434 755 333	134 074 757 785	1 801 438	874 242	[illegible]	7 [illegible] 647 284 [illegible]
1·460	[illegible]	3 [illegible]	[illegible]	3 254	[illegible]	9·126 712 941 [illegible] 3427
61	[illegible] 343 707	3 [illegible]	[illegible]	2 264	[illegible]	5 444 367 [illegible]
62	[illegible]	2 [illegible]	[illegible]	1 272	[illegible]	4 174 324 [illegible]
63	[illegible]	2 526 264 [illegible]	8 [illegible]	0 278	[illegible]	[illegible]
64	585 907 999 891	132 128 844 438	7 440 127	869 281	[illegible]	[illegible]
1·465	0·961 718 036 844 329	[illegible]	386 570 847	8 281	999	9·120 361 384 894 1076
66	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
67	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
68	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
69	[illegible]	130 204 [illegible]	[illegible]	864 262	[illegible]	[illegible]
1·470	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
71	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
72	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
73	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
74	[illegible]	128 [illegible]	[illegible]	859 [illegible]	[illegible]	[illegible]
1·475	[illegible]	7 924 154 603	377 933 371	8 166	1021	9·107 593 127 126 1520
76	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
77	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
78	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
79	[illegible]	126 [illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1·480	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
81	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
82	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
83	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
84	[illegible]	124 [illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1·485	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
86	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
87	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
88	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
89	[illegible]	122 [illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1·490	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
91	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
92	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
93	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
94	385 112 476 234	[illegible]	1 605 [illegible]	[illegible]	[illegible]	[illegible]
1·495	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
96	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
97	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
98	[illegible]	466 447 456	[illegible]	[illegible]	[illegible]	[illegible]
99	986 037 651 857	119 108 823 454	7 624 004	833 047	[illegible]	[illegible]

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2}dt$. (2) FROM $t = 1{\cdot}000$ TO $t = 3{\cdot}000$.

1·500] [1·598

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	log. $\frac{2}{\sqrt{\pi}}\epsilon^{-t^2} + 10$.
1·500	0·966 105 146 475 311	237 148 106 014	1 427 164 898	6 651 519	17 126	9·075 292 475 034 5977
2	342 294 581 325	5 727 592 635	20 513 379	34 337	182	2 684 970 965 2505
4	578 022 173 960	4 313 713 593	13 879 042	17 098	238	0 073 992 540 0482
6	812 335 887 552	2 906 451 649	07 261 944	6 599 805	293	9·067 459 539 758 9906
8	0·967 045 242 339 201		00 662 139		347	4 841 612 622 0778
		231 505 789 510		6 582 458		
1·510	0·967 276 748 128 712	0 111 709 830	1 394 079 680	65 058	17 400	9·062 220 211 129 3098
12	506 859 838 541	228 724 195 207	87 514 622	47 606	452	9·059 595 335 280 6865
14	735 584 033 749	7 343 228 191	80 967 017	30 102	504	6 966 985 076 2080
16	962 927 261 939	5 968 791 276	74 436 915	12 548	554	4 335 160 515 8473
18	0·968 188 896 053 215		67 924 367		604	1 699 861 599 6854
		224 600 866 908		6 494 944		
1·520	0·968 413 496 920 123	3 239 437 484	1 361 429 424	77 291	17 653	9·049 061 088 327 6412
22	636 736 357 607	1 884 485 351	54 952 133	59 590	701	6 418 840 699 7419
24	858 620 842 959	0 535 992 809	48 492 542	41 843	748	3 773 118 715 9873
26	0·969 079 156 835 768	219 193 942 109	42 050 700	24 049	794	1 123 922 376 3774
28	298 350 777 877		35 626 651		839	9·038 471 251 680 9124
		217 858 315 458		6 406 210		
1·530	0·969 516 209 093 336	6 529 095 017	1 329 220 441	6 388 326	17 884	9·035 815 106 629 5921
32	732 738 188 353	5 206 262 902	22 832 115	70 399	927	3 155 487 222 4166
34	947 944 451 255	3 889 801 186	16 461 717	52 428	970	0 492 393 459 3858
36	0·970 161 834 252 441	2 579 691 897	10 109 288	34 416	18 012	9·027 825 825 340 4998
38	374 413 944 338		03 774 872		053	5 155 782 865 7587
		211 275 917 026		6 316 363		
1·540	0·970 585 689 861 364	209 978 458 517	1 297 458 509	6 298 269	18 094	9·022 482 266 035 1622
42	795 668 319 880	8 687 298 277	91 160 240	80 136	133	9·019 805 274 848 7106
44	0·971 004 355 618 157	7 402 418 173	84 880 104	61 964	172	7 124 809 306 4037
46	211 758 036 330	6 123 800 034	78 618 140	43 754	210	4 440 869 408 2416
48	417 881 836 364		72 374 385		247	1 753 455 154 2243
		204 851 425 648		6 225 508		
1·550	0·971 622 733 262 013	3 585 276 771	1 266 148 878	07 225	18 283	9·009 062 566 544 3518
52	826 318 538 784	2 325 335 118	59 941 653	6 188 906	318	6 368 203 578 6240
54	0·972 028 643 873 902	1 071 582 371	53 752 747	70 553	353	3 670 366 257 0410
56	229 715 456 273	199 824 000 178	47 582 194	52 166	387	0 969 054 579 6028
58	429 539 456 450		41 430 028		420	8·998 264 268 546 3093
		198 582 570 150		6 133 746		
1·560	0·972 628 122 026 600	7 347 273 868	1 235 296 282	15 294	18 452	8·995 556 008 157 1606
62	825 469 300 469	6 118 092 881	29 180 988	6 096 810	484	2 844 273 412 1567
64	0·973 021 582 393 349	4 895 008 703	23 084 177	78 296	514	0 129 064 311 2976
66	216 482 402 053	3 678 002 822	17 005 881	59 752	544	8·987 410 380 854 5832
68	410 160 404 875		10 946 130		573	4 688 223 042 0136
		192 467 056 692		6 041 178		
1·570	0·973 602 627 461 567	1 262 151 741	1 204 904 951	22 577	18 602	8·981 962 590 873 5888
72	793 889 613 308	0 063 269 367	1 198 882 375	03 948	629	8·979 233 484 349 3088
74	983 952 882 675	188 870 390 940	92 878 427	5 985 292	656	6 500 903 469 1735
76	0·974 172 823 273 614	7 683 497 804	86 893 135	66 609	682	3 764 848 233 1831
78	360 506 771 419		80 926 526		707	1 025 318 641 3373
		186 502 571 278		5 947 902		
1·580	0·974 547 009 342 697	5 327 592 655	1 174 978 624	29 170	18 732	8·968 282 314 693 6364
82	732 336 935 351	4 158 543 201	1 169 049 453	10 415	756	5 535 836 390 0802
84	916 495 478 553	2 995 404 162	63 139 039	5 891 636	779	2 785 883 730 6689
86	0·975 099 490 882 715	1 838 156 760	57 247 403	72 835	801	0 032 456 715 4022
88	281 329 039 475		51 374 567		822	8·957 275 555 344 2804
		180 686 782 193		5 854 013		
1·590	0·975 462 015 821 668	179 541 261 638	1 145 520 554	35 170	18 843	8·954 515 179 617 3033
92	641 557 083 306	8 401 576 254	39 685 385	16 306	863	1 751 329 534 4710
94	819 958 659 560	7 267 707 175	33 869 078	5 797 424	883	8·948 984 005 095 7835
96	997 226 366 735	6 139 635 521	28 071 655	78 523	901	6 213 206 301 2408
98	0·976 173 366 002 255		22 293 132		919	3 438 933 150 8428
		175 017 342 389		5 759 603		

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t=1{\cdot}000$ TO $t=3{\cdot}000$.

1·600] [1·698

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	$\log.\ \frac{2}{\sqrt{\pi}}e^{-t^2}+10$.
1·600	0·976 348 383 344 644	173 900 808 860	1 116 533 529	5 740 667	18 936	8·940 661 185 644 5896
2	522 284 153 504	2 790 015 999	10 792 861	21 714	953	8·937 879 963 782 4812
4	695 074 169 503	1 684 944 852	05 071 147	02 746	968	5 095 267 564 5175
6	866 759 114 355	0 585 576 451	1 099 368 401	5 683 763	984	2 307 096 990 6986
8	0·977 037 344 690 806	169 491 891 813	93 684 638	5 664 765	998	8·929 515 452 061 0245
1·610	0·977 206 836 582 619	8 403 871 939	1 088 019 874	45 753	19 011	8·926 720 332 775 4952
12	375 240 454 558	7 321 497 819	82 374 120	26 729	024	3 921 739 134 1107
14	542 561 952 377	6 244 750 427	76 747 392	07 692	037	1 119 671 136 8709
16	708 806 702 804	5 173 610 728	71 139 700	5 588 644	048	8·918 314 128 783 7759
18	873 980 313 531	164 108 059 672	65 551 056	5 569 585	059	5 505 112 074 8256
1·620	0·978 038 088 373 203	3 048 078 201	1 059 981 471	50 515	19 069	8·912 692 621 010 0202
22	201 136 451 404	1 993 647 245	54 430 956	31 436	079	8·909 876 655 589 3595
24	363 130 098 650	0 944 747 726	48 899 519	12 349	088	7 057 215 812 8436
26	524 074 846 376	159 901 360 556	43 387 170	5 493 253	096	4 234 301 680 4724
28	683 976 206 932	158 863 466 638	37 893 918	5 474 149	103	1 407 913 192 2461
1·630	0·978 842 839 673 570	7 831 046 870	1 032 419 768	55 039	19 110	8 898 578 050 348 1645
32	0·979 000 670 720 440	6 804 082 140	26 964 729	35 922	117	5 744 713 148 2277
34	157 474 802 580	5 782 553 333	21 528 807	16 800	122	2 907 901 592 4356
36	313 257 355 913	4 766 441 326	16 112 007	5 397 673	127	0 067 615 680 7884
38	468 023 797 240	153 755 726 992	10 714 334	5 378 542	131	8·887 223 855 413 2859
1·640	0·979 621 779 524 232	2 750 391 200	1 005 335 792	59 407	19 135	8·884 376 620 789 9282
42	774 529 915 432	1 750 414 815	999 976 386	40 269	138	1 525 911 810 7152
44	926 280 330 247	0 755 778 698	994 636 117	21 128	140	8·878 671 728 475 6470
46	0·980 077 036 108 945	149 766 463 709	989 314 989	01 986	142	5 814 070 784 7236
48	226 802 572 654	148 782 450 707	984 013 003	5 282 843	143	2 952 938 737 9450
1·650	0·980 375 585 023 360	7 803 720 547	978 730 160	63 699	19 144	8·870 088 332 335 3112
52	523 388 743 907	6 830 254 086	973 466 461	44 555	144	8·867 220 251 576 8221
54	670 218 997 993	5 862 032 180	968 221 906	25 412	143	4 348 696 462 4778
56	816 081 030 173	4 899 035 686	962 996 494	06 270	142	1 473 666 992 2783
58	960 980 065 859	143 941 245 463	957 790 223	5 187 131	140	8·858 595 163 166 2235
1·660	0·981 104 921 311 322	2 988 642 371	952 603 093	67 993	19 137	8·855 713 184 984 3135
62	247 909 953 693	2 041 207 271	947 435 099	48 859	134	2 827 732 446 5483
64	389 951 160 964	1 098 921 031	942 286 240	29 729	130	8·849 938 805 552 9279
66	531 050 081 995	0 161 764 520	937 156 511	10 602	126	7 046 404 303 4522
68	671 211 846 515	139 229 718 611	932 045 909	5 091 481	121	4 150 528 698 1214
1·670	0·981 810 441 565 127	8 302 764 184	926 954 428	72 365	19 116	8·841 251 178 736 9352
72	948 744 329 310	7 380 882 121	921 882 062	53 255	110	8·838 348 354 419 8939
74	0·982 086 125 211 431	6 464 053 314	916 828 807	34 152	103	5 442 055 746 9973
76	222 589 264 745	5 552 258 659	911 794 655	15 056	096	2 532 282 718 2456
78	358 141 523 405	134 645 479 060	906 779 599	4 995 967	088	8·829 619 035 333 6385
1·680	0·982 492 787 002 465	3 743 695 429	901 783 632	76 887	19 080	8·826 702 313 593 1763
82	626 530 697 894	2 846 888 685	896 806 744	57 816	071	3 782 117 496 8588
84	759 377 586 578	1 955 039 756	891 848 928	38 754	062	0 858 447 044 6862
86	891 332 626 334	1 068 129 582	886 910 174	19 702	052	8·817 931 302 236 6582
88	0·983 022 400 755 916	130 186 139 110	881 990 472	4 900 660	042	5 000 683 072 7751
1·690	0·983 152 586 895 026	129 309 049 298	877 089 812	4 881 630	19 031	8·812 066 589 553 0367
92	281 895 944 324	8 436 841 115	872 208 182	62 611	019	8·809 129 021 677 4431
94	410 332 785 439	7 569 495 544	867 345 572	43 604	007	6 187 979 445 9943
96	537 902 280 983	6 706 993 575	862 501 968	24 609	18 994	3 243 462 858 6903
98	664 609 274 558	125 849 316 216	857 677 359	4 805 628	981	0 295 471 915 5310

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2}dt$. (2) FROM $t=1\cdot000$ TO $t=3\cdot000$.

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	$\log.\frac{2}{\sqrt{\pi}}\epsilon^{-t^2}+10.$
1·700	0·983 790 458 590 775	124 996 444 485	852 871 731	4 786 660	18 968	8·797 344 006 616 5165
2	915 455 035 259	4 148 359 413	48 085 071	67 706	954	·794 389 066 961 6468
4	0·984 039 603 394 673	3 305 042 048	43 317 365	48 767	939	·791 430 652 950 9218
6	162 908 436 721	2 466 473 451	38 568 598	29 844	924	·788 468 764 584 3416
8	285 374 910 172	121 632 634 696	33 838 754	4 710 935	908	·785 503 401 861 9062
1·710	0·984 407 007 544 868	0 803 506 878	829 127 819	4 692 043	18 892	8·782 534 564 783 6156
12	527 811 051 746	119 979 071 102	24 435 776	73 168	875	·779 562 253 349 4697
14	647 790 122 848	9 159 308 494	19 762 608	54 310	858	·776 586 467 559 4687
16	766 949 431 343	8 344 200 196	15 108 298	35 469	841	·773 607 207 413 6124
18	885 293 631 539	117 533 727 367	10 472 829	4 616 646	823	·770 624 472 911 9008
1·720	0·985 002 827 358 906	6 727 871 184	805 856 183	4 597 842	18 804	8·767 638 264 054 3341
22	119 555 230 090	5 926 612 843	01 258 341	79 057	785	·764 648 580 840 9121
24	235 481 842 933	5 129 933 559	796 679 284	60 292	766	·761 655 423 271 6349
26	350 611 776 492	4 337 814 567	92 118 992	41 546	746	·758 658 791 346 5024
28	464 949 591 059	113 550 237 121	87 577 446	4 522 821	725	·755 658 685 065 5148
1·730	0·985 578 499 828 180	2 767 182 496	783 054 625	04 117	18 704	8·752 655 104 428 6719
32	691 267 010 677	1 988 631 988	78 550 508	4 485 434	683	·749 648 049 435 9738
34	803 255 642 665	1 214 566 913	74 065 075	66 773	661	·746 637 520 087 4204
36	914 470 209 578	0 444 968 612	69 598 302	48 134	639	·743 623 516 383 0118
38	0·986 024 915 178 189	109 679 818 443	65 150 168	4 429 518	616	·740 606 038 322 7481
1·740	0·986 134 594 996 633	8 919 097 793	760 720 650	10 924	18 593	8·737 585 085 906 6290
42	243 514 094 426	8 162 788 067	56 309 726	4 392 355	570	·734 560 659 134 6548
44	351 676 882 493	7 410 870 696	51 917 371	73 809	546	·731 532 758 006 8253
46	459 087 753 189	6 663 327 134	47 543 562	55 288	521	·728 501 382 523 1406
48	565 751 080 323	105 920 138 860	43 188 274	4 336 792	496	·725 466 532 683 6007
1·750	0·986 671 671 219 182	5 181 287 378	738 851 482	318 320	18 471	8·722 428 208 488 2055
52	776 852 506 560	4 446 754 216	34 533 162	299 875	446	·719 386 409 936 9552
54	881 299 260 776	3 716 520 929	30 233 287	81 455	419	·716 341 137 029 8496
56	985 015 781 705	2 990 569 097	25 951 832	63 062	393	·713 292 389 766 8887
58	0·987 088 006 350 802	102 268 880 328	21 688 769	4 244 696	366	·710 240 168 148 0727
1·760	0·987 190 275 231 130	1 551 436 255	717 444 073	26 357	18 339	8·707 184 472 173 4014
62	291 826 667 385	0 838 218 540	13 217 716	08 046	311	·704 125 301 842 8749
64	392 664 885 925	0 129 208 870	09 009 670	4 189 763	283	·701 062 657 156 4932
66	492 794 094 795	99 424 388 963	04 819 907	71 508	255	·697 996 538 114 2562
68	592 218 483 758	98 723 740 564	00 648 399	4 153 282	226	·694 926 944 716 1640
1·770	0·987 690 942 224 322	98 027 245 448	696 495 117	35 085	18 197	8·691 853 876 962 2166
72	788 969 469 770	97 334 885 416	92 360 032	16 918	167	·688 777 334 852 4140
74	886 304 355 186	96 646 642 302	88 243 114	4 098 781	137	·685 697 318 386 7561
76	982 950 997 489	95 962 497 970	84 144 333	80 674	107	·682 613 827 565 2430
78	0·988 078 913 495 458	95 282 434 310	80 063 659	4 062 597	076	·679 526 862 387 8747
1·780	0·988 174 195 929 768	94 606 433 248	676 001 062	44 552	18 045	8·676 436 422 854 6512
82	268 802 363 016	93 934 476 738	71 956 510	26 538	014	·673 342 508 965 5724
84	362 736 839 755	93 266 546 766	67 929 972	08 556	17 982	·670 245 120 720 6384
86	456 003 386 520	92 602 625 349	63 921 416	3 990 606	950	·667 144 258 119 8492
88	548 606 011 870	91 942 694 538	59 930 811	3 972 688	918	·664 039 921 163 2048
1·790	0·988 640 548 706 408	91 286 736 415	655 958 123	54 803	17 885	8·660 932 109 850 7051
92	731 835 442 823	90 634 733 095	52 003 320	36 951	852	·657 820 824 182 3502
94	822 470 175 918	89 986 666 725	48 066 370	19 132	818	·654 706 064 158 1401
96	912 456 842 644	89 342 519 488	44 147 237	01 348	785	·651 587 829 778 0747
98	0·989 001 899 362 132	88 702 273 599	40 245 889	3 883 597	751	·648 466 121 042 1541

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t=1{\cdot}000$ TO $t=3{\cdot}000$.

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	log. $\frac{2}{\sqrt{\pi}}e^{-t^2}+10$.
1·800	0·989 090 501 635 731	88 065 911 307	636 362 292	3 865 881	17 716	8·645 340 937 950 3783
2	178 567 547 037	87 433 414 895	2 496 412	48 199	682	·642 212 280 502 7473
4	266 000 961 932	86 804 766 683	628 648 212	30 553	646	·639 080 148 699 2611
6	352 805 728 615	86 179 949 023	4 817 659	12 942	611	·635 944 542 539 9196
8	438 985 677 639	85 558 944 306	1 004 718	3 795 366	575	·632 805 462 024 7229
1·810	0·989 524 544 621 944	84 941 734 954	617 209 351	77 827	17 540	8·629 662 907 153 6710
12	609 486 356 899	84 328 303 430	3 431 525	60 323	503	·626 516 877 926 7638
14	693 814 660 329	83 718 632 229	609 671 201	42 857	467	·623 367 374 344 0014
16	777 533 292 557	83 112 703 884	5 928 344	25 427	430	·620 214 396 405 3838
18	860 645 996 441	82 510 500 967	2 202 918	3 708 034	393	·617 057 944 110 9110
1·820	0·989 943 156 497 408	81 912 006 083	598 494 883	3 690 679	17 355	8·613 898 017 460 5829
22	0·990 025 068 503 491	81 317 201 879	4 804 205	73 361	318	·610 734 616 454 3996
24	106 385 705 369	80 726 071 035	1 130 843	56 082	280	·607 567 741 092 3611
26	187 111 776 405	80 138 596 274	587 474 762	38 840	241	·604 397 391 374 4674
28	267 250 372 678	79 554 760 352	3 835 921	3 621 637	203	·601 223 567 300 7184
1·830	0·990 346 805 133 031	78 974 546 068	580 214 284	04 473	17 164	8·598 046 268 871 1142
32	425 779 679 099	78 397 936 258	576 609 811	3 587 348	125	·594 865 496 085 6548
34	504 177 615 357	77 824 913 796	3 022 462	70 263	086	·591 681 248 944 3402
36	582 002 529 153	77 255 461 596	569 452 199	53 217	046	·588 493 527 447 1703
38	659 257 990 749	76 689 562 614	5 898 983	3 536 210	006	·585 302 331 594 1452
1·840	0·990 735 947 553 363	76 127 199 841	562 362 772	19 244	16 966	8·582 107 661 385 2649
42	812 074 753 204	75 568 356 313	558 843 528	02 318	926	·578 909 516 820 5293
44	887 643 109 517	75 013 015 103	5 341 210	3 485 433	885	·575 707 897 899 9386
46	962 656 124 620	74 461 159 327	1 855 777	68 589	844	·572 502 804 623 4926
48	0·991 037 117 283 947	73 912 772 139	548 387 188	51 785	803	·569 294 236 991 1913
1·850	0·991 111 030 056 086	73 367 836 736	544 935 403	3 435 023	16 762	8·566 082 195 003 0349
52	184 397 892 822	72 826 336 356	1 500 380	18 302	721	·562 866 678 659 0232
54	257 224 229 178	72 288 254 279	538 082 077	01 624	679	·559 647 687 959 1563
56	329 512 483 456	71 753 573 825	4 680 454	3 384 987	637	·556 425 222 903 4342
58	401 266 057 282	71 222 278 358	1 295 467	3 368 392	595	·553 199 283 491 8568
1·860	0·991 472 488 335 640	70 694 351 283	527 927 075	51 840	16 552	8·549 969 869 724 4242
62	543 182 686 923	70 169 776 047	4 575 235	35 330	510	·546 736 981 601 1364
64	613 352 462 970	69 648 536 142	1 239 906	18 863	467	·543 500 619 121 9934
66	683 000 999 112	69 130 615 099	517 921 043	02 439	424	·540 260 782 286 9951
68	752 131 614 211	68 615 996 496	4 618 604	3 286 058	381	·537 017 471 096 1417
1·870	0·991 820 747 610 707	68 104 663 951	511 332 545	69 721	16 337	8·533 770 685 549 4330
72	888 852 274 657	67 596 601 127	508 062 824	53 428	294	·530 520 425 646 8690
74	956 448 875 784	67 091 791 731	4 809 396	37 178	250	·527 266 691 388 4499
76	0·992 023 540 667 515	66 590 219 512	1 572 218	20 972	206	·524 009 482 774 1755
78	090 130 887 027	66 091 868 266	498 351 246	3 204 811	162	·520 748 799 804 0458
1·880	0·992 156 222 755 294	65 596 721 831	495 146 435	3 188 694	16 117	8·517 484 642 478 0610
82	221 819 477 125	65 104 764 089	1 957 742	72 621	073	·514 217 010 796 2209
84	286 924 241 214	64 615 978 969	488 785 121	56 593	028	·510 945 904 758 5256
86	351 540 220 183	64 130 350 441	5 628 528	40 610	15 983	·507 671 324 364 9751
88	415 670 570 624	63 647 862 524	2 487 917	3 124 672	938	·504 393 269 615 5694
1·890	0·992 479 318 433 148	63 168 499 279	479 363 245	08 780	15 893	8·501 111 740 510 3084
92	542 486 932 427	62 692 244 814	6 254 465	3 092 933	847	·497 826 737 049 1922
94	605 179 177 241	62 219 083 282	3 161 532	77 131	802	·494 538 259 232 2208
96	667 398 260 522	61 748 998 880	0 084 401	61 375	756	·491 246 307 059 3941
98	729 147 259 403	61 281 975 855	467 023 026	3 045 666	710	·487 950 880 530 7123

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t=1·000$ TO $t=3·000$.

1·900] [1·998

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	$\log\frac{2}{\sqrt{\pi}}e^{-t^2}+10.$
1·900	0·992 790 429 235 257		463 977 360		15 664	8·484 651 979 646 1752
		60 817 998 495		3 030 002		
2	851 247 233 752		0 947 358		617	·481 349 604 405 7828
		60 357 051 137		14 384		
4	911 604 284 889		457 932 974		571	·478 043 754 809 5353
		59 899 118 163		2 998 813		
6	971 503 403 052		4 934 161		525	·474 734 430 857 4325
		59 444 184 002		83 288		
8	0·993 030 947 587 054		1 950 873		478	·471 421 632 549 4745
		58 992 233 129		2 967 810		
1·910	0·993 089 939 820 183		448 983 063		15 431	8·468 105 359 885 6613
		58 543 250 067		52 379		
12	148 483 070 250		6 030 684		384	·464 785 612 865 9928
		58 097 219 383		36 994		
14	206 580 289 633		3 093 689		337	·461 462 391 490 4691
		57 654 125 694		21 657		
16	264 234 415 327		0 172 032		290	·458 135 695 759 0902
		57 213 953 662		06 367		
18	321 448 368 988		437 265 665		243	·454 805 525 671 8561
		56 776 687 996		2 891 124		
1·920	0·993 378 225 056 985		434 374 541		15 195	8·451 471 881 228 7667
		56 342 313 455		75 929		
22	434 567 370 440		1 498 612		148	·448 134 762 429 8221
		55 910 814 843		60 781		
24	490 478 185 284		428 637 831		100	·444 794 169 275 0223
		55 482 177 012		45 681		
26	545 960 362 296		5 792 151		052	·441 450 101 764 3673
		55 056 384 861		30 628		
28	601 016 747 157		2 961 522		004	·438 102 559 897 8570
		54 633 423 339		2 815 624		
1·930	0·993 655 650 170 496		420 145 898		14 956	8·434 751 543 675 4915
		54 213 277 441		00 667		
32	709 863 447 937		417 345 231		908	·431 397 053 097 2708
		53 795 932 210		2 785 759		
34	763 659 380 148		4 559 472		860	·428 039 088 163 1949
		53 381 372 739		70 899		
36	817 040 752 886		1 788 573		812	·424 677 648 873 2637
		52 969 584 166		56 087		
38	870 010 337 052		409 032 486		763	·421 312 735 227 4773
		52 560 551 680		2 741 324		
1·940	0·993 922 570 888 733		406 291 162		14 715	8·417 944 347 225 8357
		52 154 260 519		26 609		
42	974 725 149 251		3 564 553		666	·414 572 484 868 3388
		51 750 695 966		11 943		
44	0·994 026 475 845 217		0 852 610		618	·411 197 148 154 9868
		51 349 843 356		2 697 325		
46	077 825 688 574		398 155 285		569	·407 818 337 085 7795
		50 951 688 072		82 757		
48	128 777 376 645		5 472 528		520	·404 436 051 660 7169
		50 556 215 544		2 668 237		
1·950	0·994 179 333 592 189		392 804 291		14 471	8·401 050 291 879 7992
		50 163 411 253		2 653 766		
52	229 497 003 442		0 150 525		422	·397 661 057 743 0262
		49 773 260 728		39 344		
54	279 270 264 169		387 511 181		373	·394 268 349 250 3980
		49 385 749 546		24 971		
56	328 656 013 716		4 886 210		323	·390 872 166 401 9146
		49 000 863 337		10 648		
58	377 656 877 053		2 275 562		274	·387 472 509 197 5759
		48 618 587 775		2 596 374		
1·960	0·994 426 275 464 828		379 679 188		14 225	8·384 069 377 637 3820
		48 238 908 587		82 149		
62	474 514 373 415		7 097 039		175	·380 662 771 721 3329
		47 861 811 548		67 973		
64	522 376 184 964		4 529 065		126	·377 252 691 449 4286
		47 487 282 483		53 848		
66	569 863 467 457		1 975 218		076	·373 839 136 821 6690
		47 115 307 265		39 771		
68	616 978 774 712		369 435 447		027	·370 422 107 838 0543
		46 745 871 818		2 525 744		
1·970	0·994 663 724 646 530		366 909 703		13 977	8·367 001 604 498 5843
		46 378 962 116		11 767		
72	710 103 608 646		4 397 935		927	·363 577 626 803 2590
		46 014 564 180		2 497 840		
74	756 118 172 826		1 900 096		878	·360 150 174 752 0785
		45 652 664 085		83 962		
76	801 770 836 911		359 416 134		828	·356 719 248 345 0429
		45 293 247 951		70 134		
78	847 064 084 862		6 945 999		778	·353 284 847 582 1519
		44 936 301 952		2 456 356		
1·980	0·994 892 000 386 814		354 489 643		13 728	8·349 846 972 463 4058
		44 581 812 309		42 628		
82	936 582 199 122		2 047 015		678	·346 405 622 988 8044
		44 229 765 294		28 950		
84	980 811 964 416		349 618 065		628	·342 960 799 158 3478
		43 880 147 229		15 322		
86	0·995 024 692 111 644		7 202 743		578	·339 512 500 972 0360
		43 532 944 486		01 744		
88	068 225 056 130		4 800 999		528	·336 060 728 429 8690
		43 188 143 487		2 388 216		
1·990	0·995 111 413 199 617		342 412 783		13 478	8·332 605 481 531 8467
		42 845 730 704		74 738		
92	154 258 930 321		0 038 045		428	·329 146 760 277 9692
		42 505 692 659		61 310		
94	196 764 622 980		337 676 735		378	·325 684 564 668 2365
		42 168 015 924		47 933		
96	238 932 638 904		5 328 802		327	·322 218 894 702 6485
		41 832 687 123		34 605		
98	280 765 226 026		2 994 196		277	·318 749 750 381 2054
		41 499 692 926		2 321 328		

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	log. $\frac{2}{\sqrt{\pi}}e^{-t^2}+10.$ −
2·000	0·995 322 265 018 953	41 169 020 058	330 672 868	2 308 101	13 227	8·315 277 131 703 9070
2	363 434 039 011	40 840 655 292	328 364 766	294 925	177	·311 801 038 670 7533
4	404 274 694 303	514 585 450	6 069 842	281 799	126	·308 321 471 281 7445
6	444 789 279 754	190 797 408	3 788 043	268 723	076	·304 838 429 536 8804
8	484 980 077 161	39 869 278 087	1 519 320	2 255 697	026	·301 351 913 436 1611
2·010	0·995 524 849 355 248	550 014 464	319 263 623	242 722	12 975	8·297 861 922 979 5866
12	564 399 369 712	232 993 562	7 020 902	229 797	925	·294 368 458 167 1568
14	603 632 363 274	38 918 202 457	4 791 105	216 922	875	·290 871 518 998 8718
16	642 550 565 731	605 628 273	2 574 183	204 098	824	·287 371 105 474 7316
18	681 156 194 004	38 295 258 188	0 370 086	2 191 324	774	·283 867 217 594 7362
2·020	0·995 719 451 452 192	37 987 079 426	308 178 762	178 600	12 724	8·280 359 855 358 8855
22	757 438 531 618	681 079 264	6 000 162	165 927	673	·276 849 018 767 1796
24	795 119 610 882	377 245 030	3 834 235	153 304	623	·273 334 707 819 6185
26	832 496 855 912	075 564 099	1 680 930	140 732	572	·269 816 922 516 2022
28	869 572 420 011	36 776 023 901	299 540 199	2 128 210	522	·266 295 662 856 9306
2·030	0·995 906 348 443 912	478 611 912	297 411 989	115 738	12 472	8·262 770 928 841 8038
32	942 827 055 824	183 315 661	5 296 251	103 316	421	·259 242 720 470 8218
34	979 010 371 484	35 890 122 726	3 192 935	2 090 945	371	·255 711 037 743 9845
36	0·996 014 900 494 210	599 020 737	1 101 989	078 625	321	·252 175 880 661 2921
38	050 499 514 947	35 309 997 372	289 023 365	2 066 354	270	·248 637 249 222 7444
2·040	0·996 085 809 512 320	023 040 362	286 957 010	054 134	12 220	8·245 095 143 428 3415
42	120 832 552 681	34 738 137 485	4 902 876	041 964	170	·241 549 563 278 0833
44	155 570 690 167	455 276 573	2 860 912	029 844	120	·238 000 508 771 9699
46	190 025 966 740	174 445 505	0 831 068	017 775	069	·234 447 979 910 0013
48	224 200 412 245	33 895 632 212	278 813 293	2 005 756	019	·230 891 976 692 1775
2·050	0·996 258 096 044 457	618 824 674	276 807 538	1 993 786	11 969	8·227 332 499 118 4985
52	291 714 869 131	344 010 923	4 813 751	981 867	919	·223 769 547 188 9642
54	325 058 880 054	071 179 039	2 831 884	969 999	869	·220 203 120 903 5747
56	358 130 059 093	32 800 317 154	0 861 885	958 180	819	·216 633 220 262 3300
58	390 930 376 247	32 531 413 449	268 903 705	1 946 411	769	·213 059 845 265 2300
2·060	0·996 423 461 789 696	264 456 154	266 957 294	934 692	11 719	8·209 482 995 912 2748
62	455 726 245 850	31 999 433 552	5 022 602	923 024	669	·205 902 672 203 4644
64	487 725 679 403	736 333 974	3 099 579	911 405	619	·202 318 874 138 7988
66	519 462 013 377	475 145 800	1 188 174	1 899 836	569	·198 731 601 718 2779
68	550 937 159 177	31 215 857 462	259 288 338	1 888 317	519	·195 140 854 941 9018
2·070	0·996 582 153 016 638	30 958 457 440	257 400 022	876 847	11 469	8·191 546 633 809 6705
72	613 111 474 079	702 934 266	5 523 174	865 428	420	·187 948 938 321 5840
74	643 814 408 345	449 276 520	3 657 746	854 058	370	·184 347 768 477 6422
76	674 263 684 865	197 472 831	1 803 688	842 738	320	·180 743 124 277 8452
78	704 461 157 696	29 947 511 881	249 960 951	1 831 467	271	·177 135 005 722 1930
2·080	0·996 734 408 669 576	699 382 397	248 129 483	820 246	11 221	8·173 523 412 810 6856
82	764 108 051 974	453 073 160	6 309 237	809 075	171	·169 908 345 543 3229
84	793 561 125 133	208 572 998	4 500 162	1 797 953	122	·166 289 803 920 1050
86	822 769 698 131	28 965 870 788	2 702 210	786 880	073	·162 667 787 941 0319
88	851 735 568 919	28 724 955 459	0 915 329	1 775 857	023	·159 042 297 606 1035
2·090	0·996 880 460 524 378	485 815 986	239 139 472	764 883	10 974	8·155 413 332 915 3200
92	908 946 340 364	248 441 397	7 374 589	753 958	925	·151 780 893 868 6812
94	937 194 781 761	012 820 765	5 620 631	743 083	876	·148 144 980 466 1871
96	965 207 602 526	27 778 943 217	3 877 549	732 256	827	·144 505 592 707 8379
98	992 986 545 743	27 546 797 924	2 145 293	1 721 479	777	·140 862 730 593 6334

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t = 1{\cdot}000$ TO $t = 3{\cdot}000$.

2·100] [2·198

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	$\log \frac{2}{\sqrt{\pi}}e^{-t^2} + 10.$
2·100	0·997 020 533 343 667	27 316 374 110	230 423 814	1 710 750	10 729	8·137 216 394 123 5737
2	047 849 717 777	087 661 046	228 713 064	700 071	679	·133 566 583 297 6588
4	074 937 378 823	26 860 648 053	7 012 993	1 689 440	631	·129 913 298 115 8886
6	101 798 026 876	635 324 499	5 323 554	678 858	582	·126 256 538 578 2632
8	128 433 351 375	26 411 679 803	3 644 696	1 668 324	533	·122 596 304 684 7826
2·110	0·997 154 845 031 178	189 703 431	221 976 372	657 840	10 485	8·118 932 596 435 4468
12	181 034 734 609	25 969 384 900	0 318 532	647 403	436	·115 265 413 830 2557
14	207 004 119 509	750 713 771	218 671 128	637 016	388	·111 594 756 869 2095
16	232 754 833 280	533 679 658	7 034 113	626 676	339	·107 920 625 552 3080
18	258 288 512 938	25 318 272 222	5 407 436	1 616 385	291	·104 243 019 879 5512
2·120	0·997 283 606 785 161	104 481 172	213 791 051	606 143	10 243	8·100 561 939 850 9393
22	308 711 266 332	24 892 296 264	212 184 908	595 948	10 195	·096 877 385 466 4721
24	333 603 562 596	681 707 304	0 588 960	585 801	147	·093 189 356 726 1497
26	358 285 269 899	472 704 145	209 003 159	575 703	099	·089 497 853 629 9720
28	382 757 974 044	24 265 276 689	7 427 456	1 565 652	051	·085 802 876 177 9391
2·130	0·997 407 023 250 733	059 414 886	205 861 804	555 649	10 003	8·082 104 424 370 0511
32	431 082 665 619	23 855 108 731	4 306 154	545 694	9 955	·078 402 498 206 3077
34	454 937 774 350	652 348 271	2 760 460	535 787	908	·074 697 097 686 7092
36	478 590 122 621	451 123 598	1 224 674	525 927	860	·070 988 222 811 2554
38	502 041 246 219	23 251 424 851	199 698 747	1 516 114	813	·067 275 873 579 9464
2·140	0·997 525 292 671 070	053 242 218	198 182 633	506 349	9 765	8·063 560 049 992 7822
42	548 345 913 288	22 856 565 934	6 676 284	496 631	718	·059 840 752 049 7627
44	571 202 479 222	661 386 281	5 179 653	486 960	671	·056 117 979 750 8881
46	593 863 865 503	467 693 589	3 692 692	477 337	624	·052 391 733 096 1582
48	616 331 559 092	22 275 478 233	2 215 356	1 467 760	577	·048 662 012 085 5730
2·150	0·997 638 607 037 325	084 730 638	190 747 595	458 230	9 530	8·044 928 816 719 1327
52	660 691 767 963	21 895 441 273	189 289 365	448 748	483	·041 192 146 996 8371
54	682 587 209 237	707 600 656	7 840 617	439 311	436	·037 452 002 918 6863
56	704 294 809 893	521 199 350	6 401 306	429 922	390	·033 708 384 484 6803
58	725 816 009 242	21 336 227 965	4 971 384	1 420 578	343	·029 961 291 694 8190
2·160	0·997 747 152 237 208	152 677 159	183 550 806	411 282	9 297	8·026 210 724 549 1025
62	768 304 914 367	20 970 537 635	2 139 524	402 031	250	·022 456 683 047 5308
64	789 275 452 002	789 800 142	0 737 493	1 392 827	204	·018 699 167 190 1039
66	810 065 252 145	610 455 476	179 344 666	383 669	158	·014 938 176 976 8217
68	830 675 707 621	20 432 494 479	7 960 997	1 374 557	112	·011 173 712 407 6843
2·170	0·997 851 108 202 100	255 908 039	176 586 441	365 490	9 066	8·007 405 773 482 6917
72	871 364 110 139	080 687 089	5 220 950	356 470	021	·003 634 360 201 8439
74	891 444 797 227	19 906 822 608	3 864 481	347 495	8 975	7·999 859 472 565 1408
76	911 351 619 835	734 305 622	2 516 986	338 565	929	·996 081 110 572 5825
78	931 085 925 457	19 563 127 201	1 178 421	1 329 681	884	·992 299 274 224 1690
2·180	0·997 950 649 052 659	393 278 462	169 848 739	320 843	8 839	7·988 513 963 519 9003
82	970 042 331 121	224 750 566	8 527 896	312 049	793	·984 725 178 459 7763
84	989 267 081 687	057 534 719	7 215 847	303 301	748	·980 932 919 043 7971
86	0·998 008 324 616 405	18 891 622 172	5 912 546	1 294 598	703	·977 137 185 271 9627
88	027 216 238 578	18 727 004 224	4 617 949	1 285 939	658	·973 337 977 144 2731
2·190	0·998 045 943 242 801	563 672 214	163 332 009	277 325	8 614	7·969 535 294 660 7282
92	064 506 915 016	401 617 530	2 054 684	268 756	569	·965 729 137 821 3281
94	082 908 532 546	240 831 603	0 785 928	260 232	525	·961 919 506 626 0728
96	101 149 364 149	081 305 907	159 525 696	251 752	480	·958 106 401 074 9622
98	119 230 670 056	17 923 031 963	8 273 944	1 243 316	436	·954 289 821 167 9964

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t=1·000$ TO $t=3·000$.

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	$\log.\frac{2}{\sqrt{\pi}}e^{-t^2}+10.$
2·200	0·998 137 153 702 018		157 030 628		8 392	7·950 469 766 905 1754
		17 766 001 334		1 234 924		
2	154 919 703 352		5 795 704		348	·946 646 238 286 4992
		610 205 630		226 576		
4	172 529 908 982		4 569 128		304	·942 819 235 311 9677
		455 636 502		218 273		
6	189 985 545 484		3 350 855		260	·938 988 757 981 5811
		302 285 647		210 013		
8	207 287 831 130		2 140 842		216	·935 154 806 295 3392
		17 150 144 804		1 201 797		
2·210	0·998 224 437 975 934		150 939 046		8 173	7·931 317 380 253 2420
		16 999 205 758		1 193 624		
12	241 437 181 693		149 745 422		129	·927 476 479 855 2897
		849 460 336		185 495		
14	258 286 642 029		8 559 927		086	·923 632 105 101 4821
		700 900 409		177 409		
16	274 987 542 437		7 382 519		043	·919 784 255 991 8193
		553 517 890		169 366		
18	291 541 060 327		6 213 153		8 000	·915 932 932 526 3012
		16 407 304 737		1 161 366		
2·220	0·998 307 948 365 065		145 051 786		7 957	7·912 078 134 704 9280
		262 252 951		153 410		
22	324 210 618 016		3 898 377		914	·908 219 862 527 6995
		118 354 575		145 496		
24	340 328 972 590		2 752 881		871	·904 358 115 994 6158
		15 975 601 694		137 625		
26	356 304 574 284		1 615 256		829	·900 492 895 105 6768
		833 986 437		129 796		
28	372 138 560 721		0 485 460		786	·896 624 199 860 8826
		15 693 500 977		1 122 010		
2·230	0·998 387 832 061 698		139 363 451		7 744	7·892 752 030 260 2333
		554 137 526		114 266		
32	403 386 199 224		8 249 185		702	·888 876 386 303 7286
		415 888 341		106 564		
34	418 802 087 565		7 142 621		660	·884 997 267 991 3688
		278 745 720		098 904		
36	434 080 833 286		6 043 717		618	·881 114 675 323 1537
		142 702 004		091 287		
38	449 223 535 289		4 952 430		576	·877 228 608 299 0834
		15 007 749 573		1 083 711		
2·240	0·998 464 231 284 863		133 868 719		7 534	7·873 339 066 919 1579
		14 873 880 854		076 176		
42	479 105 165 717		2 792 543		493	·869 446 051 183 3771
		741 088 311		068 683		
44	493 846 254 027		1 723 860		451	·865 549 561 091 7412
		609 364 451		061 232		
46	508 455 618 478		0 662 628		410	·861 649 596 644 2500
		478 701 823		053 822		
48	522 934 320 302		129 608 806		369	·857 746 157 840 9035
		14 349 093 017		1 046 453		
2·250	0·998 537 283 413 319		128 562 353		7 328	7·853 839 244 681 7019
		220 530 664		039 125		
52	551 503 943 983		7 523 228		287	·849 928 857 166 6450
		093 007 436		031 838		
54	565 596 951 419		6 491 391		246	·846 014 995 295 7329
		13 966 516 045		024 591		
56	579 563 467 464		5 466 800		206	·842 097 659 068 9656
		841 049 246		017 385		
58	593 404 516 710		4 449 414		165	·838 176 848 486 3430
		13 716 599 832		1 010 220		
2·260	0·998 607 121 116 542		123 439 194		7 125	7·834 252 563 547 8652
		593 160 637		1 003 095		
62	620 714 277 179		2 436 100		084	·830 324 804 253 5322
		470 724 538		996 010		
64	634 185 001 716		1 440 090		045	·826 393 570 603 3440
		349 284 448		988 965		
66	647 534 286 164		0 451 125		005	.822 458 862 597 3005
		228 833 323		981 960		
68	660 763 119 487		119 469 165		6 965	·818 520 680 235 4018
		13 109 364 158		974 995		
2·270	0·998 673 872 483 645		118 494 170		6 926	7·814 579 023 517 6479
		12 990 869 988		968 069		
72	686 863 353 634		7 526 101		886	·810 633 892 444 0388
		873 343 888		961 183		
74	699 736 697 521		6 564 917		847	·806 685 287 014 5744
		756 778 970		954 337		
76	712 493 476 492		5 610 581		807	·802 733 207 229 2548
		641 168 390		947 529		
78	725 134 644 881		4 663 052		768	·798 777 653 088 0800
		12 526 505 338		940 761		
2·280	0·998 737 661 150 219		113 722 291		6 729	7·794 818 624 591 0500
		412 783 047		934 031		
82	750 073 933 266		2 788 260		691	·790 856 121 738 1647
		299 994 787		927 341		
84	762 373 928 053		1 860 919		652	·786 890 144 529 4242
		188 133 868		920 689		
86	774 562 061 921		0 940 230		613	·782 920 692 964 8285
		077 193 638		914 075		
88	786 639 255 559		0 026 155		575	·778 947 767 044 3775
		11 967 167 482		907 500		
2·290	0·998 798 606 423 041		109 118 655		6 537	7·774 971 366 768 0714
		858 048 828		900 963		
92	810 464 471 869		8 217 692		499	·770 991 492 135 9100
		749 831 136		894 465		
94	822 214 303 005		7 323 227		461	·767 008 143 147 8933
		642 507 909		888 004		
96	833 856 810 914		6 435 223		423	·763 021 319 804 0215
		536 072 686		881 581		
98	845 392 883 599		5 553 642		385	·759 031 022 104 2944
		11 430 519 044		875 196		

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t=1·000$ TO $t=3·000$.

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	log. $\frac{2}{\sqrt{\pi}}e^{-t^2}+10$.
2·300	0·998 856 823 402 643	11 325 840 598	104 678 446	868 848	6 348	7·755 037 250 048 7121
2	868 149 243 241	222 031 000	3 809 598	62 538	310	·751 040 003 637 2746
4	879 371 274 241	119 083 939	2 947 060	56 265	273	·747 039 282 869 9818
6	890 490 358 180	016 993 144	2 090 796	50 029	236	·743 035 087 746 8338
8	901 507 351 323	10 915 752 377	1 240 767	843 830	199	·739 027 418 267 8306
2·310	0·998 912 423 103 700	815 355 440	100 396 937	37 668	6 162	7·735 016 274 432 9722
12	923 238 459 140	715 796 171	99 559 269	31 542	125	·731 001 656 242 2585
14	933 954 255 311	617 068 444	8 727 727	25 453	089	·726 983 563 695 6896
16	944 571 323 755	519 166 171	7 902 273	19 401	052	·722 961 996 793 2655
18	955 090 489 926	10 422 083 298	7 082 872	813 385	016	·718 936 955 534 9862
2·320	0·998 965 512 573 224	325 813 811	96 269 488	07 405	5 980	7·714 908 439 920 8516
22	975 838 387 035	230 351 728	5 462 083	01 461	944	·710 876 449 950 8618
24	986 068 738 763	135 691 106	4 660 622	795 552	908	·706 840 985 625 0168
26	996 204 429 869	041 826 036	3 865 070	89 680	873	·702 802 046 943 3166
28	0·999 006 246 255 904	9 948 750 645	3 075 390	783 843	837	·698 759 633 905 7611
2·330	0·999 016 195 006 550	856 459 097	92 291 548	78 041	5 802	7·694 713 746 512 3504
32	026 051 465 647	764 945 591	1 513 507	72 275	766	·690 664 384 763 0845
34	035 816 411 238	674 204 358	0 741 232	66 543	731	·686 611 548 657 9634
36	045 490 615 596	584 229 669	89 974 689	60 847	696	·682 555 238 196 9870
38	055 074 845 265	9 495 015 827	9 213 842	755 185	661	·678 495 453 380 1554
2·340	0·999 064 569 861 092	406 557 170	88 458 657	49 559	5 627	7·674 432 194 207 4686
42	073 976 418 262	318 848 072	7 709 098	43 966	592	·670 365 460 678 9265
44	083 295 266 334	231 882 940	6 965 132	38 408	558	·666 295 252 794 5292
46	092 527 149 274	145 656 216	6 226 723	32 885	524	·662 221 570 554 2767
48	101 672 805 490	9 060 162 378	5 493 839	727 395	490	·658 144 413 958 1690
2·350	0·999 110 732 967 868	8 975 395 934	84 766 443	21 940	5 456	7·654 063 783 006 2061
52	119 708 363 802	891 351 431	4 044 504	16 518	422	·649 979 677 698 3879
54	128 599 715 232	808 023 445	3 327 986	11 130	388	·645 892 098 034 7145
56	137 407 738 677	725 406 589	2 616 856	05 775	355	·641 801 044 015 1858
58	146 133 145 266	8 643 495 509	1 911 080	700 454	321	·637 706 515 639 8020
2·360	0·999 154 776 640 775	562 284 882	81 210 626	695 166	5 288	7·633 608 512 908 5629
62	163 338 925 658	481 769 423	80 515 460	89 911	255	·629 507 035 821 4686
64	171 820 695 080	401 943 874	79 825 548	84 690	222	·625 402 084 378 5190
66	180 222 638 954	322 803 015	9 140 859	79 501	189	·621 293 658 579 7143
68	188 545 441 969	8 244 341 657	8 461 358	674 344	156	·617 181 758 425 0543
2·370	0·999 196 789 783 626	166 554 643	77 787 014	69 220	5 124	7·613 066 383 914 5391
72	204 956 338 269	089 436 849	7 117 794	64 129	091	·608 947 535 048 1686
74	213 045 775 119	012 983 184	6 453 665	59 070	059	·604 825 211 825 9430
76	221 058 758 303	7 937 188 589	5 794 595	54 042	027	·600 699 414 247 8621
78	228 995 946 892	7 862 048 036	5 140 553	649 047	4 995	·596 570 142 313 9260
2·880	0·999 236 857 994 929	787 556 531	74 491 505	44 084	4 963	7·592 437 396 024 1346
82	244 645 551 460	713 709 109	3 847 422	39 152	932	·588 301 175 378 4880
84	252 359 260 569	640 500 840	3 208 269	34 252	900	·584 161 480 376 9862
86	259 999 761 409	567 926 822	2 574 018	29 383	869	·580 018 311 019 6292
88	267 567 688 231	7 495 982 188	1 944 634	624 545	838	·575 871 667 306 4170
2·390	0·999 275 063 670 419	424 662 099	71 320 089	19 739	4 807	7·571 721 549 237 3495
92	282 488 332 518	353 961 749	0 700 350	14 963	776	·567 567 956 812 4268
94	289 842 294 267	283 876 362	0 085 387	10 219	745	·563 410 890 031 6489
96	297 126 170 629	214 401 194	69 475 168	05 505	714	·559 250 348 895 0157
98	304 340 571 824	7 145 531 531	8 869 663	600 821	684	·555 086 333 402 5273

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t = 1·000$ TO $t = 3·000$.

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	log. $\frac{2}{\sqrt{\pi}}e^{-t^2}$ + 10.
2·400	0·999 311 486 103 355		68 268 842		4 653	7·550 918 843 554 1837
		7 077 262 689		596 168		
2	318 563 366 044		7 672 675		623	·546 747 879 349 9849
		009 590 014		91 545		
4	325 572 956 057		7 081 130		593	·542 573 440 789 9309
		6 942 508 884		86 952		
6	332 515 464 941		6 494 178		563	·538 395 527 874 0216
		876 014 706		82 389		
8	339 391 479 648		5 911 789		533	·534 214 140 602 2571
		6 810 102 917		577 856		
2·410	0·999 346 201 582 565		65 333 933		4 503	7·530 029 278 974 6373
		744 768 984		73 352		
12	352 946 351 549		4 760 581		474	·525 840 942 991 1624
		680 008 403		68 879		
14	359 626 359 952		4 191 702		445	·521 649 132 651 8322
		615 816 701		64 434		
16	366 242 176 654		3 627 268		415	·517 453 847 956 6468
		552 189 433		60 019		
18	372 794 366 087		3 067 249		386	·513 255 088 905 6061
		6 489 122 184		555 633		
2·420	0·999 379 283 488 271		62 511 617		4 357	7·509 052 855 498 7103
		426 610 567		51 275		
22	385 710 098 838		1 960 341		328	·504 847 147 735 9592
		364 650 226		46 947		
24	392 074 749 064		1 413 394		300	·500 637 965 617 3529
		303 236 832		42 647		
26	398 377 985 896		0 870 747		271	·496 425 309 142 8913
		242 366 085		38 376		
28	404 620 351 981		0 332 371		243	·492 209 178 312 5745
		6 182 033 714		534 133		
2·430	0·999 410 802 385 694		59 798 238		4 214	7·487 989 573 126 4026
		122 235 476		29 919		
32	416 924 621 170		9 268 319		186	·483 766 493 584 3753
		062 967 157		25 733		
34	422 987 588 328		8 742 586		158	·479 539 939 686 4929
		004 224 571		21 574		
36	428 991 812 899		8 221 012		131	·475 309 911 432 7552
		5 946 003 559		17 444		
38	434 937 816 458		7 703 568		103	·471 076 408 823 1623
		5 888 299 991		513 341		
2·440	0·999 440 826 116 449		57 190 227		4 075	7·466 839 431 857 7142
		831 109 763		09 266		
42	446 657 226 212		6 680 962		048	·462 598 980 536 4108
		774 428 801		05 218		
44	452 431 655 013		6 175 744		021	·458 355 054 859 2523
		718 253 057		01 197		
46	458 149 908 071		5 674 547		3 993	·454 107 654 826 2385
		662 578 511		497 204		
48	463 812 486 581		5 177 343		966	·449 856 780 437 3694
		5 607 401 168		493 238		
2·450	0·999 469 419 887 749		54 684 105		3 939	7·445 602 431 692 6452
		552 717 063		489 298		
52	474 972 604 811		4 194 807		913	·441 344 608 592 0657
		498 522 255		85 385		
54	480 471 127 067		3 709 422		886	·437 083 311 135 6310
		444 812 834		81 499		
56	485 915 939 901		3 227 922		860	·432 818 539 323 3411
		391 584 912		477 640		
58	491 307 524 813		2 750 282		833	·428 550 293 155 1959
		5 338 834 629		473 807		
2·460	0·999 496 646 359 442		52 276 476		3 807	7·424 278 572 631 1955
		286 558 154		469 999		
62	501 932 917 596		1 806 476		781	·420 003 377 751 3399
		234 751 677		66 218		
64	507 167 669 273		1 340 258		755	·415 724 708 515 6291
		183 411 419		62 463		
66	512 351 080 692		0 877 795		729	·411 442 564 924 0630
		132 533 625		458 734		
68	517 483 614 317		0 419 060		704	·407 156 946 976 6417
		5 082 114 564		455 031		
2·470	0·999 522 565 728 881		49 964 030		3 678	7·402 867 854 673 3652
		032 150 535		51 352		
72	527 597 879 416		9 512 677		653	·398 575 288 014 2335
		4 982 637 857		447 700		
74	532 580 517 273		9 064 978		627	·394 279 246 999 2465
		933 572 880		44 072		
76	537 514 090 152		8 620 905		602	·389 979 731 628 4043
		884 951 974		40 470		
78	542 399 042 127		8 180 435		577	·385 676 741 901 7069
		4 836 771 539		436 893		
2·480	0·999 547 235 813 666		47 743 542		3 552	7·381 370 277 819 1542
		789 027 997		33 340		
82	552 024 841 663		7 310 202		528	·377 060 339 380 7464
		741 717 795		429 813		
84	556 766 559 458		6 880 389		503	·372 746 926 586 4833
		694 837 406		26 310		
86	561 461 396 863		6 454 080		479	·368 430 039 436 3649
		648 383 326		22 831		
88	566 109 780 189		6 031 249		454	·364 109 677 930 3914
		4 602 352 077		419 377		
2·490	0·999 570 712 132 266		45 611 872		3 430	7·359 785 842 068 5626
		556 740 205		15 947		
92	575 268 872 471		5 195 926		406	·355 468 531 850 8786
		511 544 279		12 541		
94	579 780 416 751		4 783 385		382	·351 127 747 277 3394
		466 760 895		409 158		
96	584 247 177 645		4 374 226		358	·346 793 488 347 9449
		422 386 668		05 800		
98	588 669 564 313		3 968 426		335	·342 455 755 062 6953
		4 378 418 242		402 466		

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t = 1\text{·}000$ TO $t = 3\text{·}000$.

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	$\log \frac{2}{\sqrt{\pi}} e^{-t^2} + 10$.
2·500	0·999 593 047 982 555	4 334 852 281	43 565 961	399 155	3 311	7·338 114 547 421 5903
2	597 382 834 836	291 685 475	166 806	5 867	3 288	·333 769 865 424 6302
4	601 674 520 311	248 914 537	42 770 939	2 603	64	·329 421 709 071 8149
6	605 923 434 848	206 536 201	378 336	389 362	41	·325 070 078 363 1443
8	610 129 971 049	4 164 547 227	41 988 974	386 144	18	·320 714 973 298 6185
2·510	0·999 614 294 518 275	122 944 396	41 602 831	2 948	3 195	7·316 356 393 878 2374
12	618 417 462 671	081 724 513	219 882	379 776	72	·311 994 340 102 0012
14	622 499 187 184	040 884 407	40 840 106	6 626	50	·307 628 811 969 9097
16	626 540 071 592	000 420 927	463 480	3 499	27	·303 259 809 481 9630
18	630 540 492 519	3 960 330 946	089 981	370 394	05	·298 887 332 638 1610
2·520	0·999 634 500 823 465	920 611 360	39 719 586	367 312	3 082	7·294 511 381 438 5039
22	638 421 434 825	881 259 085	352 275	4 252	60	·290 131 955 882 9915
24	642 302 693 911	842 271 062	38 988 023	1 213	38	·285 749 055 971 6239
26	646 144 964 973	803 644 253	626 810	358 197	16	·281 362 681 704 4010
28	649 948 609 225	3 765 375 640	268 613	355 202	2 995	·276 972 833 081 3229
2·530	0·999 653 713 984 865	727 462 229	37 913 411	2 229	2 973	7·272 579 510 102 3896
32	657 441 447 094	689 901 047	561 182	349 278	51	·268 182 712 767 6011
34	661 131 348 141	652 689 144	211 904	6 348	30	·263 782 441 076 9574
36	664 784 037 284	615 823 588	36 865 556	3 439	09	·259 378 695 030 4584
38	668 399 860 872	3 579 301 471	522 117	340 552	2 888	·254 971 474 628 1042
2·540	0·999 671 979 162 343	543 119 906	36 181 565	337 685	2 867	7·250 560 779 869 8948
42	675 522 282 249	507 276 025	35 843 880	4 839	46	·246 146 610 755 8301
44	679 029 558 274	471 766 985	509 041	2 015	25	·241 728 967 285 9102
46	682 501 325 259	436 589 959	177 026	329 211	04	·237 307 849 460 1351
48	685 937 915 218	3 401 742 143	34 847 815	326 427	2 784	·232 883 257 278 5048
2·550	0·999 689 339 657 361	367 220 755	34 521 388	323 664	2 763	7·228 455 190 741 0193
52	692 706 878 115	333 023 030	197 725	0 921	43	·224 023 649 847 6785
54	696 039 901 146	299 146 227	33 876 803	318 199	23	·219 588 634 598 4825
56	699 339 047 372	265 587 622	558 605	5 496	02	·215 150 144 993 4312
58	702 604 634 995	3 232 344 513	243 109	312 814	2 683	·210 708 181 032 5248
2·560	0·999 705 836 979 508	199 414 218	32 930 295	0 151	2 663	7·206 262 742 715 7631
62	709 036 393 727	166 794 074	620 144	307 508	43	·201 813 830 043 1462
64	712 203 187 801	134 481 438	312 636	4 885	23	·197 361 443 014 6740
66	715 337 669 239	102 473 687	007 751	2 281	04	·192 905 581 630 3467
68	718 440 142 926	3 070 768 216	31 705 471	299 696	2 584	·188 446 245 890 1641
2·570	0·999 721 510 911 143	039 362 442	31 405 774	7 131	2 565	7·183 983 435 794 1263
72	724 550 273 585	008 253 800	108 643	4 585	46	·179 517 151 342 2332
74	727 558 527 385	2 977 439 742	30 814 058	2 058	27	·175 047 392 534 4850
76	730 535 967 127	946 917 743	521 999	289 550	08	·170 574 159 370 8815
78	733 482 884 869	2 916 685 293	232 449	287 061	2 489	·166 097 451 851 4228
2·580	0·999 736 399 570 163	886 739 905	29 945 388	4 590	2 471	7·161 617 269 976 1088
82	739 286 310 068	857 079 107	660 798	2 138	52	·157 133 613 744 9396
84	742 143 389 175	827 700 448	378 660	279 705	34	·152 646 483 157 9152
86	744 971 089 623	798 601 493	098 955	7 290	15	·148 155 878 215 0356
88	747 769 691 116	2 769 779 828	28 821 665	274 893	2 397	·143 661 798 916 3008
2·590	0·999 750 539 470 943	741 233 055	28 546 772	2 514	2 379	7·139 164 245 261 7107
92	753 280 703 999	712 958 797	274 258	0 153	61	·134 663 217 251 2654
94	755 993 662 795	684 954 692	004 105	267 810	43	·130 158 714 884 9649
96	758 678 617 487	657 218 397	27 736 295	5 485	25	·125 650 738 162 8091
98	761 335 835 884	2 629 747 587	470 810	263 178	07	·121 139 287 084 7981

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t=1·000$ TO $t=3·000$.

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	$\log \frac{2}{\sqrt{\pi}}e^{-t^2}+10$.
2·600	0·999 763 965 583 471		27 207 632		2 290	7·116 624 361 650 9319
		2 602 539 955		260 888		
2	66 568 123 426		26 946 744		72	·112 105 961 861 2105
		575 593 211		258 616		
4	69 143 716 637		688 128		55	·107 584 087 715 6338
		548 905 083		6 361		
6	71 692 621 720		431 767		38	·103 058 739 214 2019
		522 473 316		4 123		
8	74 215 095 036		177 644		21	·098 529 916 356 9148
		2 496 295 672		251 903		
2·610	0·999 776 711 390 708		25 925 742		2 204	7·093 997 619 143 7725
		470 369 930		249 699		
12	79 181 760 638		676 043		2 187	·089 461 847 574 7749
		444 693 887		7 512		
14	81 626 454 526		428 530		70	·084 922 601 649 9221
		419 265 357		5 343		
16	84 045 719 883		183 188		53	·080 379 881 369 2141
		394 082 169		3 190		
18	86 439 802 052		24 939 998		36	·075 833 686 732 6509
		2 369 142 171		241 053		
2·620	0·999 788 808 944 224		24 698 945		2 120	7·071 284 017 740 2324
		344 443 226		238 933		
22	91 153 387 450		460 012		04	·066 730 874 391 9587
		319 983 215		6 830		
24	93 473 370 665		223 182		2 087	·062 174 256 687 8298
		295 760 033		4 742		
26	95 769 130 697		23 988 440		71	·057 614 164 627 8457
		271 771 593		2 671		
28	98 040 902 290		755 768		55	·053 050 598 212 0063
		2 248 015 825		230 617		
2·630	0·999 800 288 918 115		23 525 152		2 039	7·048 483 557 440 3117
		224 890 673		228 578		
32	02 513 408 789		296 574		23	·043 913 042 312 7619
		201 194 100		6 555		
34	04 714 602 888		070 019		07	·039 339 052 829 3568
		178 124 080		4 547		
36	06 892 726 969		22 845 472		1 992	·034 761 588 990 0966
		155 278 608		2 556		
38	09 048 005 577		622 916		76	·030 180 650 794 9811
		2 132 655 692		220 580		
2·640	0·999 811 180 661 268		22 402 337		1 961	7·025 596 238 244 0103
		110 253 356		218 619		
42	13 290 914 625		183 718		45	·021 008 351 337 1844
		088 069 638		6 674		
44	15 378 984 263		21 967 044		30	·016 416 990 074 5032
		066 102 595		4 744		
46	17 445 086 857		752 299		15	·011 822 154 455 9668
		044 350 295		2 830		
48	19 489 437 153		539 470		00	·007 223 844 481 5752
		2 022 810 826		210 930		
2·650	0·999 821 512 247 978		21 328 540		1 885	7·002 622 060 151 3283
		001 482 286		209 045		
52	23 513 730 264		119 494		70	6·998 016 801 465 2262
		1 980 362 792		7 176		
54	25 494 093 056		20 912 319		55	·993 408 068 423 2689
		959 450 473		5 321		
56	27 453 543 529		706 998		40	·988 795 861 025 4564
		938 743 475		3 481		
58	29 392 287 004		503 517		26	·984 180 179 271 7886
		1 918 239 958		201 655		
2·660	0·999 831 310 526 963		20 301 862		1 811	6·979 561 023 162 2656
		1 897 938 096		199 844		
62	33 208 465 059		102 018		1 797	·974 938 392 696 8874
		877 836 078		8 047		
64	35 086 301 136		19 903 971		82	·970 312 287 875 6540
		857 932 107		6 265		
66	36 944 233 243		707 707		68	·965 682 708 698 5653
		838 224 400		4 496		
68	38 782 457 643		513 210		54	·961 049 655 165 6214
		1 818 711 190		192 742		
2·670	0·999 840 601 168 833		19 320 468		1 740	6·956 413 127 276 8223
		1 799 390 722		1 002		
72	42 400 559 555		129 465		26	·951 773 125 032 1680
		780 261 257		189 276		
74	44 180 820 812		18 940 189		12	·947 129 648 431 6584
		761 321 068		7 564		
76	45 942 141 880		752 625		1 699	·942 482 697 475 2936
		742 568 443		5 865		
78	47 684 710 322		566 760		85	·937 832 272 163 0736
		1 724 001 683		184 181		
2·680	0·999 849 408 712 006		18 382 579		1 671	6·933 178 372 494 9983
		705 619 104		2 509		
82	51 114 331 110		200 070		58	·928 520 998 471 0679
		687 419 034		0 851		
84	52 801 750 144		019 218		44	·923 860 150 091 2822
		669 399 816		179 207		
86	54 471 149 959		17 840 012		31	·919 195 827 355 6412
		651 559 804		7 576		
88	56 122 709 763		662 436		18	·914 528 030 264 1451
		1 633 897 368		175 958		
2·690	0·999 857 756 607 132		17 486 478		1 605	6·909 856 758 816 7937
		616 410 890		4 353		
92	59 373 018 021		312 125		1 592	·905 182 013 013 5871
		599 098 765		2 761		
94	60 972 116 787		139 364		79	·900 503 792 854 5253
		581 959 401		1 182		
96	62 554 076 188		16 968 182		66	·895 822 098 339 6082
		564 991 219		169 616		
98	64 119 067 407		798 566		53	·891 136 929 468 8360
		1 548 192 653		168 063		

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t = 1·000$ TO $t = 3·000$.

2·700] [2·798

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	$\log \frac{2}{\sqrt{\pi}}e^{-t^2} + 10.$
2·700	0·999 865 667 260 059		16 630 503		1 541	6·886 448 286 242 2084
		1 531 562 150		166 522		
2	7 198 822 209		463 981		28	·881 756 168 659 7257
		15 098 169		4 994		
4	8 713 920 378		298 987		16	·877 060 576 721 3878
		1 498 799 182		3 478		
6	0·999 870 212 719 559		135 509		03	·872 361 510 427 1946
		182 663 673		1 975		
8	1 695 383 232		15 973 533		1 491	·867 658 969 777 1462
		1 466 690 139		160 484		
2·710	0·999 873 162 073 372		15 813 049		1 479	6·862 952 954 771 2425
		50 877 090		159 006		
12	4 612 950 462		654 043		66	·858 243 465 409 4837
		35 223 047		7 539		
14	6 048 173 509		496 504		54	·853 530 501 691 8696
		19 726 543		6 085		
16	7 467 900 053		340 419		42	·848 814 063 618 4003
		04 386 124		4 643		
18	8 872 286 177		185 776		31	·844 094 151 189 0757
		1 389 200 348		153 212		
2·720	0·999 880 261 486 525		15 032 564		1 419	6·839 370 764 403 8959
		74 167 784		1 793		
22	1 635 654 309		14 880 771		07	·834 643 903 262 8610
		59 287 013		0 386		
24	2 994 941 322		730 385		1 395	·829 913 567 765 9707
		44 556 628		148 991		
26	4 339 497 950		581 394		84	·825 179 757 913 2253
		29 975 235		7 607		
28	5 669 473 185		433 786		72	·820 442 473 704 6246
		1 315 541 449		146 235		
2·730	0·999 886 985 014 633		14 287 551		1 361	6·815 701 715 140 1687
		01 253 897		4 874		
32	8 286 268 531		142 677		50	·810 957 482 219 8576
		1 287 111 221		3 525		
34	9 573 379 751		13 999 152		38	·806 209 774 943 6913
		73 112 068		2 186		
36	0·999 890 846 491 820		856 966		27	·801 458 593 311 6697
		59 255 103		0 859		
38	2 105 746 923		716 106		16	·796 703 937 323 7929
		1 245 538 997		139 543		
2·740	0·999 893 351 285 919		13 576 563		1 305	6·791 945 806 980 0608
		31 962 434		8 238		
42	4 583 248 353		438 325		1 294	·787 184 202 280 4736
		18 524 109		6 944		
44	5 801 772 462		301 380		83	·782 419 123 225 0311
		05 222 729		5 661		
46	7 006 995 191		165 719		72	·777 650 569 813 7334
		1 192 057 010		4 389		
48	8 199 052 201		031 331		62	·772 878 542 046 5805
		1 179 025 679		133 127		
2·750	0·999 899 378 077 880		12 898 204		1 251	6·768 103 039 923 5723
		66 127 476		1 876		
52	0·999 900 544 205 356		766 328		41	·763 324 063 444 7089
		53 361 148		0 635		
54	1 697 566 503		635 693		30	·758 541 612 609 9903
		40 725 455		129 405		
56	2 838 291 958		506 288		20	·753 755 687 419 4165
		28 219 167		8 185		
58	3 966 511 125		378 102		09	·748 966 287 872 9874
		1 115 841 065		126 976		
2·760	0·999 905 082 352 190		12 251 127		1 199	6·744 173 413 970 7031
		03 589 938		5 777		
62	6 185 942 128		125 350		89	·739 377 065 712 5636
		1 091 464 588		4 588		
64	7 277 406 716		000 762		79	·734 577 243 098 5689
		79 463 826		3 409		
66	8 356 870 542		11 877 353		69	·729 773 946 128 7189
		67 586 473		2 240		
68	9 424 457 015		755 113		59	·724 967 174 803 0137
		1 055 831 360		121 081		
2·770	0·999 910 480 288 375		11 634 032		1 149	6·720 156 929 121 4533
		44 197 328		119 932		
72	1 524 485 703		514 100		39	·715 343 209 084 0377
		32 683 228		8 793		
74	2 557 168 930		395 307		30	·710 526 014 690 7668
		21 287 920		7 663		
76	3 578 456 851		277 644		20	·705 705 345 941 6407
		10 010 276		6 543		
78	4 588 467 127		161 101		10	·700 881 202 836 6594
		998 849 175		115 433		
2·780	0·999 915 587 316 302		11 045 668		1 101	6·696 053 585 375 8228
		987 803 507		4 332		
82	6 575 119 809		10 931 335		91	·691 222 493 559 1311
		976 872 172		3 241		
84	7 551 991 981		818 094		82	·686 387 927 386 5841
		966 054 078		2 159		
86	8 518 046 059		705 935		73	·681 549 886 858 1818
		955 348 142		1 086		
88	9 473 394 201		594 849		63	·676 708 371 973 9244
		944 753 294		110 023		
2·790	0·999 920 418 147 495		10 484 826		1 054	6·671 863 382 733 8117
		934 268 468		108 969		
92	1 352 415 963		375 857		45	·667 014 919 137 8438
		923 892 611		7 924		
94	2 276 308 574		267 933		36	·662 162 981 186 0207
		913 624 678		6 888		
96	3 189 933 252		161 045		27	·657 307 568 878 3423
		903 463 633		5 861		
98	4 093 396 886		055 185		18	·652 448 682 214 8087
		893 408 449		104 842		

TABLE OF THE VALUES OF $H = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t = 1·000$ TO $t = 3·000$.

2·800] [2·898

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	$\log \frac{2}{\sqrt{\pi}}e^{-t^2} + 10.$
2·800	0·999 924 986 805 335	883 458 107	9 950 342	103 833	1 009	6·647 586 321 195 4199
2	5 870 263 441	873 611 598	846 509	102 832	1 001	·642 720 485 820 1759
4	6 743 875 039	863 867 921	743 677	101 841	992	·637 851 176 089 0766
6	7 607 742 960	854 226 085	641 836	100 857	983	·632 978 392 002 1222
8	8 461 969 046	844 685 107	540 979	99 883	975	·628 102 133 559 3124
2·810	0·999 929 306 654 152	835 244 011	9 441 096	98 916	966	6·623 222 400 760 6475
12	0·999 930 141 898 163	825 901 831	342 179	97 959	958	·618 339 193 606 1273
14	0 967 799 994	816 657 611	244 221	97 009	949	·613 452 512 095 7520
16	1 784 457 605	807 510 399	147 211	96 068	941	·608 562 356 229 5213
18	2 591 968 004	798 459 256	051 143	95 136	933	·603 668 726 007 4355
2·820	0·999 933 390 427 260	789 503 248	8 956 008	94 211	925	6·598 771 621 429 4944
22	4 179 930 508	780 641 452	861 797	93 294	916	·593 871 042 495 6981
24	4 960 571 960	771 872 949	768 502	92 386	908	·588 966 989 206 0466
26	5 732 444 909	763 196 833	676 116	91 486	900	·584 059 461 560 5399
28	6 495 641 743	754 612 203	584 631	90 593	892	·579 148 459 559 1779
2·830	0·999 937 250 253 945	746 118 165	8 494 038	89 709	885	6·574 233 983 201 9607
32	7 996 372 110	737 713 836	404 329	88 832	877	·569 316 032 488 8883
34	8 734 085 946	729 398 339	315 497	87 963	869	·564 394 607 419 9606
36	9 463 484 285	721 170 804	227 534	87 101	861	·559 469 707 995 1778
38	0·999 940 184 655 090	713 030 371	140 433	86 248	854	·554 541 334 214 5397
2·840	0·999 940 897 685 461	704 976 186	8 054 185	85 402	846	6·549 609 486 078 0463
42	1 602 661 647	697 007 403	7 968 784	84 563	839	·544 674 163 585 6978
44	2 299 669 050	689 123 182	884 220	83 732	831	·539 735 366 737 4940
46	2 988 792 232	681 322 694	800 488	82 908	824	·534 793 095 533 4350
48	3 670 114 925	673 605 113	717 580	82 092	816	·529 847 349 973 5208
2·850	0·999 944 343 720 039	665 969 625	7 635 488	81 283	809	6·524 898 130 057 7513
52	5 009 689 664	658 415 420	554 205	80 481	802	·519 945 435 786 1266
54	5 668 105 084	650 941 696	473 724	79 686	795	·514 989 267 158 6467
56	6 319 046 779	643 547 658	394 038	78 899	788	·510 029 624 175 3116
58	6 962 594 437	636 232 519	315 139	78 118	780	·505 066 506 836 1212
2·860	0·999 947 598 826 956	628 995 498	7 237 021	77 345	773	6·500 099 915 141 0756
62	8 227 822 454	621 835 823	159 676	76 578	766	·495 129 849 090 1748
64	8 849 658 277	614 752 726	083 097	75 819	760	·490 156 308 683 4188
66	9 464 411 002	607 745 448	007 278	75 066	753	·485 179 293 920 8075
68	0·999 950 072 156 450	600 813 236	6 932 212	74 320	746	·480 198 804 802 3410
2·870	0·999 950 672 969 686	593 955 344	6 857 892	73 581	739	6·475 214 841 328 0193
72	1 266 925 030	587 171 033	784 311	72 848	733	·470 227 403 497 8423
74	1 854 096 063	580 459 571	711 462	72 123	726	·465 236 491 311 8102
76	2 434 555 633	573 820 231	639 340	71 403	719	·460 242 104 769 9228
78	3 008 375 864	567 252 294	567 937	70 690	713	·455 244 243 872 1802
2·880	0·999 953 575 628 159	560 755 048	6 497 246	69 984	706	6·450 242 908 618 5823
82	4 136 383 207	554 327 786	427 262	69 284	700	·445 238 099 009 1292
84	4 690 710 994	547 969 809	357 978	68 591	694	·440 229 815 043 8209
86	5 238 680 802	541 680 422	289 387	67 903	687	·435 218 056 722 6574
88	5 780 361 224	535 458 938	221 484	67 223	681	·430 202 824 045 6387
2·890	0·999 956 315 820 162	529 304 677	6 154 261	66 548	675	6·425 184 117 012 7647
92	6 845 124 839	523 216 964	087 713	65 879	669	·420 161 935 624 0355
94	7 368 341 802	517 195 130	021 834	65 217	662	·415 136 279 879 4510
96	7 885 536 932	511 238 513	5 956 617	64 561	656	·410 107 149 779 0114
98	8 396 775 445	505 346 456	892 057	63 910	650	·405 074 545 322 7165

TABLE OF THE VALUES OF $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$. (2) FROM $t=1·000$ TO $t=3·000$.

t	H	Δ_1 +	Δ_2 −	Δ_3 +	Δ_4 −	$\log \frac{2}{\sqrt{\pi}}e^{-t^2}+10.$
2·900	0·999 958 902 121 901		5 828 146		644	6·400 038 466 510 5664
		499 518 310		63 266		
2	9 401 640 210		764 881		638	·394 998 913 342 5611
		493 753 429		62 627		
4	9 895 393 639		702 253		633	·389 955 885 818 7005
		488 051 176		61 995		
6	0·999 960 383 444 815		640 258		627	·384 909 383 938 9847
		482 410 917		61 368		
8	0 865 855 732		578 890		621	·379 859 407 703 4137
		476 832 027		60 747		
2·910	0·999 961 342 687 760		5 518 143		615	6·374 805 957 111 9875
		471 313 884		60 132		
12	1 814 001 643		458 011		610	·369 749 032 164 7060
		465 855 873		59 522		
14	2 279 857 516		398 489		604	·364 688 632 861 5693
		460 457 384		58 918		
16	2 740 314 899		339 570		598	·359 624 759 202 5774
		455 117 813		58 320		
18	3 195 432 713		281 250		593	·354 557 411 187 7303
		449 836 563		57 727		
2·920	0·999 963 645 269 275		5 223 523		587	6·349 486 588 817 0279
		444 613 040		57 140		
22	4 089 882 316		166 383		582	·344 412 292 090 4703
		439 446 657		56 558		
24	4 529 328 973		109 824		576	·339 334 521 008 0575
		434 336 833		55 982		
26	4 963 665 806		053 842		571	·334 253 275 569 7894
		429 282 991		55 411		
28	5 392 948 797		4 998 431		566	·329 168 555 775 6662
		424 284 560		54 845		
2·930	0·999 965 817 233 357		4 943 586		560	6·324 080 361 625 6877
		419 340 974		54 285		
32	6 236 574 332		889 301		555	·318 988 693 119 8539
		414 451 673		53 730		
34	6 651 026 005		835 571		550	·313 893 550 258 1650
		409 616 102		53 180		
36	7 060 642 108		782 391		545	·308 794 933 040 6208
		404 833 711		52 635		
38	7 465 475 819		729 756		540	·303 692 841 467 2214
		400 103 955		52 095		
2·940	0·999 967 865 579 774		4 677 661		535	6·298 587 275 537 9668
		395 426 294		51 561		
42	8 261 006 068		626 100		530	·293 478 235 252 8569
		390 800 194		51 031		
44	8 651 806 263		575 069		525	·288 365 720 611 8918
		386 225 125		50 506		
46	9 038 031 388		524 563		520	·283 249 731 615 0715
		381 700 562		49 987		
48	9 419 731 950		474 576		515	·278 130 268 262 3960
		377 225 986		49 472		
2·950	0·999 969 796 957 936		4 425 104		510	6·273 007 330 553 8652
		372 800 882		48 962		
52	0·999 970 169 758 818		376 142		505	·267 880 918 489 4793
		368 424 740		48 457		
54	0 538 183 557		327 686		500	·262 751 032 069 2380
		364 097 054		47 956		
56	0 902 280 611		279 729		496	·257 617 671 293 1416
		359 817 325		47 461		
58	1 262 097 936		232 268		491	·252 480 836 161 1899
		355 585 057		46 970		
2·960	0·999 971 617 682 993		4 185 298		486	6·247 340 526 673 3831
		351 399 758		46 484		
62	1 969 082 751		138 815		482	·242 196 742 829 7209
		347 260 943		46 002		
64	2 316 343 695		092 813		477	·237 049 484 630 2036
		343 168 131		45 525		
66	2 659 511 825		047 288		473	·231 898 752 074 8310
		339 120 843		45 052		
68	2 998 632 668		002 236		468	·226 744 545 163 6032
		335 118 607		44 584		
2·970	0·999 973 333 751 275		3 957 651		464	6·221 586 863 896 5202
		331 160 955		44 121		
72	3 664 912 230		913 531		459	·216 425 708 273 5820
		327 247 425		43 662		
74	3 992 159 655		869 869		455	·211 261 078 294 7885
		323 377 556		43 207		
76	4 315 537 210		826 662		450	·206 092 973 960 1398
		319 550 893		42 756		
78	4 635 088 104		783 906		446	·200 921 395 269 6359
		315 766 987		42 310		
2·980	0·999 974 950 855 091		3 741 596		442	6·195 746 342 223 2767
		312 025 391		41 868		
82	5 262 880 482		699 727		438	·190 567 814 821 0624
		308 325 664		41 431		
84	5 571 206 146		658 297		433	·185 385 813 062 9928
		304 667 367		40 997		
86	5 875 873 514		617 299		429	·180 200 336 949 0679
		301 050 068		40 568		
88	6 176 923 582		576 731		425	·175 011 386 479 2879
		297 473 337		40 143		
2·990	0·999 976 474 396 919		3 536 588		421	6·169 818 961 653 6526
		293 936 750		39 722		
92	6 768 333 669		496 866		417	·164 623 062 472 1621
		290 439 884		39 305		
94	7 058 773 553		457 561		413	·159 423 688 934 8164
		286 982 323		38 892		
96	7 345 755 876		418 668		409	·154 220 841 041 6154
		283 563 655		38 483		
98	7 629 319 531		380 185		405	·149 014 518 792 5592
		280 183 470		38 078		
3·000	0·999 977 909 503 001		3 342 107		401	6·143 804 722 187 6478

(3) Table of Values of $H=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$, and $G=\int_t^\infty e^{-t^2}dt$, from $t=3·0$ to $t=6·0$.

L denotes the value of Laplace's continued fraction (§9).

t	H	L	e^{-t^2}	G
3·0	0·999 977 909 503 001	·951 813 839 183 927	·000 123 409 804 087	·000 019 577 193 237
3·1	988 351 342 633	·954 514 373 156 224	·000 067 054 824 303	·000 010 323 353 804
3·2	993 974 238 848	·957 000 847 840 583	·000 035 712 849 642	·000 005 340 191 779
3·3	996 942 290 204	·959 294 708 327 178	·000 018 643 742 332	·000 002 709 824 752
3·4	998 478 006 638	·961 414 842 914 146	*·(5)9 540 162 873 079	·000 001 348 831 499
3·5	0·999 999 258 901 628	·963 377 932 668 129	·(5)4 785 117 392 129	·000 000 658 553 786
3·6	644 137 007	·965 198 747 506 567	·(5)2 352 575 200 010	315 375 366
3·7	832 848 942	·966 890 397 828 628	·(5)1 133 727 138 748	148 133 768
3·8	922 996 073	·968 464 548 822 273	·(5)0 535 534 780 279	068 242 954
3·9	965 207 751	·969 931 604 907 714	·(5)0 247 959 601 805	030 833 828
4·0	0·999 999 984 582 742	·971 300 864 958 029	·(5)0 112 535 174 719	·000 000 013 663 189
4·1	932 999 724	·972 580 654 473 280	·(5)0 050 062 180 208	005 937 745
4·2	997 144 506	·973 778 466 897 719	·(5)0 021 829 577 951	002 530 616
4·3	998 806 528	·974 901 013 211 320	·(8)9 330 287 574 505	001 057 687
4·4	999 510 829	·975 954 360 776 017	·(8)3 908 938 434 265	000 433 517
4·5	0·999 999 999 803 384	·976 943 983 556 604	·(8)1 605 228 055 186	·000 000 000 174 246
4·6	922 504	·977 874 833 415 583	·(8)0 646 143 177 311	068 679
4·7	980 048	·978 719 571 814 619	·(8)0 254 938 188 039	026 544
4·8	988 648	·979 577 750 614 522	·(8)0 098 595 055 760	010 061
4·9	995 781	·980 357 595 871 849	·(8)0 037 375 713 279	003 739
5·0	0·999 999 999 998 463	·981 094 307 287 316	·(8)0 013 887 943 865	·000 000 000 001 363
5·5	999 993	·984 229 800 386 619	·(12) 072 877 240 958	·(8)0 000 006 520 723
6·0	1—·(15) 021 516 075	·986 653 109 231 165	·(15) 231 952 283 024	·(8)0 000 000 019 069
∞	1·0	1·0	·0	0·0

* The figures in parentheses indicate the number of ciphers between the decimal point and the figures that follow. The value of H for $t=6$ is 0·999 999 999 999 999 978 483 925.

ERRATA.

Page 257, last line, *for* $\int_a$ *read* $\int_0$

" 258, line 4, *for* $\int_t^\infty e - \rho dt$, *read* $\int_t^\infty e^{-t^2}dt$

" 261, note †, *for* 1·283791 670, etc., *read* 1·128 397 167 0, etc., twice.

" 263, note *, *for* Probabilities, *read* Probabilities,"

" 266, note †, *for* Mr W. T. B., *read* Mr W. S. B.

" 271, line 3, *for* Δ_8), *read* Δ^8)

" 273, line 14, *for* + etc. $\}$. *read* + etc. $\}$.*

" 276, line 22, *for* $e-\rho$, *read* e^{-t^2}

X.—*The Relations between the Coaxial Minors of a Determinant of the Fourth Order.* By Thomas Muir, LL.D.

(Read January 31, 1898.)

1. The existence of relations between the coaxial minors of a determinant was first discovered by MacMahon in 1893. The whole literature of the subject is comprised in three papers, viz.:—

MacMahon, *Phil. Trans.*, clxxxv. pp. 111–160.
Muir, *Phil. Mag.*, 5th series, xli. pp. 537–541.
Nanson, *Phil. Mag.*, 5th series, xliv. pp. 362–367.

My present object is to continue the investigation of the relations in question, and more particularly to draw attention to an *explicit* expression for a determinant of the 4th order in terms of its own coaxial minors. At the outset some fresh considerations regarding determinants in general will be found useful.

2. As is well known, the coaxial minors of a determinant of the nth order are $2^n - 1$ in number, the determinant itself and each of the elements of its primary diagonal being counted. For example, the coaxial minors of $|a_1b_2c_3d_4|$ are

$$\begin{array}{llllll} |a_1b_2c_3d_4|, \\ |a_1b_2c_3|, & |a_1b_2d_4|, & |a_1c_3d_4|, & |b_2c_3d_4|, \\ |a_1b_2|, & |a_1c_3|, & |a_1d_4|, & |b_2c_3|, & |b_2d_4|, & |c_3d_4|, \\ a_1, & b_2, & c_3, & d_4. \end{array}$$

Of these the first $2^n - 1 - n$ may be devertebrated, if we may say so, by substituting zeros for the elements of their primary diagonals; and the determinants thus resulting are found to be of considerable interest. They appear in Cayley's well-known expansion-theorem, which for a determinant of the 3rd order is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} \cdot & a_2 & a_3 \\ b_1 & \cdot & b_3 \\ c_1 & c_2 & \cdot \end{vmatrix} + a_1 \begin{vmatrix} \cdot & b_3 \\ c_2 & \cdot \end{vmatrix} + b_2 \begin{vmatrix} \cdot & a_3 \\ c_1 & \cdot \end{vmatrix} + c_3 \begin{vmatrix} \cdot & a_2 \\ b_1 & \cdot \end{vmatrix} + a_1b_2c_3.$$

Indeed this theorem may be described as giving an expression for a determinant in terms of its own devertebrated coaxial minors and its primary diagonal elements.

Now, if we use Cayley's expansion in connection with each of the first $2^n - 1 - n$ coaxial minors, we obtain $2^n - 1 - n$ equations, linear in respect to the devertebrated minors. So that, on solving for the latter, there must result an expression for each

devertebrated coaxial minor in terms of the vertebrate coaxial minors and the primary diagonal elements. The general theorem thus obtained is

$$\begin{vmatrix} \cdot & a_2 & a_3 & a_4 & \cdots \\ b_1 & \cdot & b_3 & b_4 & \cdots \\ c_1 & c_2 & \cdot & c_4 & \cdots \\ d_1 & d_2 & d_3 & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{vmatrix} = |a_1b_2c_3d_4\ldots\ldots| - \Sigma a_1|b_2c_3d_4\ldots\ldots| + \Sigma a_1b_2|c_3d_4\ldots\ldots| + \ldots\ldots \qquad \text{(A)}$$

It may be viewed as a sort of converse of CAYLEY'S, which in outward form it very closely resembles.

3. The truth of it may be established by proceeding in the manner just indicated; but there is another available process which has the advantage of presenting it merely as the ultimate case of a more general theorem, viz., a theorem for similarly expanding a determinant which is only *partially* devertebrated.

Taking determinants of the 3rd order, we have in succession and without any difficulty of verification,

$$\begin{vmatrix} \cdot & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = |a_1b_2c_3| - a_1|b_2c_3|,$$

$$\begin{vmatrix} \cdot & a_2 & a_3 \\ b_1 & \cdot & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = |a_1b_2c_3| - a_1|b_2c_3| - b_2|a_1c_3| + a_1b_2c_3,$$

$$\begin{vmatrix} \cdot & a_2 & a_3 \\ b_1 & \cdot & b_3 \\ c_1 & c_2 & \cdot \end{vmatrix} = |a_1b_2c_3| - a_1|b_2c_3| - b_2|a_1c_3| - c_3|a_1b_2| + 2a_1b_2c_3.$$

Proceeding to the 4th order, we have with equal simplicity in the first case

$$\begin{vmatrix} \cdot & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = |a_1b_2c_3d_4| - a_1|b_2c_3d_4|. \qquad (\mathrm{A}_1)$$

For the next case we have similarly

$$\begin{vmatrix} \cdot & a_2 & a_3 & a_4 \\ b_1 & \cdot & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = \begin{vmatrix} \cdot & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} - b_2\begin{vmatrix} \cdot & a_3 & a_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix},$$

and as each of the determinants on the right has already been expanded in the new form, there is at once obtained by substitution

$$|a_1b_2c_3d_4| - a_1|b_2c_3d_4| - b_2|a_1c_3d_4| + a_1b_2|c_3d_4|. \qquad (\mathrm{A}_2$$

Again, we have

$$\begin{vmatrix} \cdot & a_2 & a_3 & a_4 \\ b_1 & \cdot & b_3 & b_4 \\ c_1 & c_2 & \cdot & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = \begin{vmatrix} \cdot & a_2 & a_3 & a_4 \\ b_1 & \cdot & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} - c_3 \begin{vmatrix} \cdot & a_2 & a_4 \\ b_1 & \cdot & b_4 \\ d_1 & d_2 & d_4 \end{vmatrix},$$

$$\begin{aligned} &= |a_1b_2c_3d_4| - a_1|b_2c_3d_4| - b_2|a_1c_3d_4| + a_1b_2|c_3d_4| \\ &\qquad - c_3\{|a_1b_2d_4| - a_1|b_2d_4| - b_2|a_1d_4| + a_1b_2d_4\}, \\ &= |a_1b_2c_3d_4| - a_1|b_2c_3d_4| - b_2|a_1c_3d_4| - c_3|a_1b_2d_4| \\ &\qquad + a_1b_2|c_3d_4| + a_1c_3|b_2d_4| + b_2c_3|a_1d_4| \\ &\qquad - a_1b_2c_3d_4. \qquad\qquad (A_3) \end{aligned}$$

And lastly, by proceeding in exactly the same way, we have the theorem of the preceding section, viz.:—

$$\begin{vmatrix} \cdot & a_2 & a_3 & a_4 \\ b_1 & \cdot & b_3 & b_4 \\ c_1 & c_2 & \cdot & c_4 \\ d_1 & d_2 & d_3 & \cdot \end{vmatrix} = |a_1b_2c_3d_4| - \Sigma a_1|b_2c_3d_4| + \Sigma a_1b_2|c_3d_4| - 3a_1b_2c_3d_4,$$

where the Σ refers to combinations of the four elements, a_1, b_2, c_3, d_4.

4. MacMahon's problem of expressing the determinant of the 4th order in terms of its coaxial minors may thus be transformed into something apparently simpler, viz., expressing the determinant in terms of its *devertebrated* coaxial minors and the primary diagonal elements.

In the case of the determinant $|a_1b_2c_3d_4|$ the eleven (*i.e.*, $2^4 - 1 - 4$) devertebrated coaxial minors are

$$\begin{vmatrix} \cdot & a_2 & a_3 & a_4 \\ b_1 & \cdot & b_3 & b_4 \\ c_1 & c_2 & \cdot & c_4 \\ d_1 & d_2 & d_3 & \cdot \end{vmatrix} \quad i.e., \quad a_2b_1c_4d_3 + a_3b_4c_1d_2 + a_4b_3c_2d_1 - \left\{ \begin{matrix} a_2b_4c_1d_3 + a_3b_1c_4d_2 \\ + a_2b_3c_4d_1 + a_4b_1c_2d_3 \\ + a_4b_3c_1d_2 + a_3b_4c_2d_1 \end{matrix} \right\} = \mathrm{D} \text{ say},$$

$$\begin{vmatrix} \cdot & a_2 & a_3 \\ b_1 & \cdot & b_3 \\ c_1 & c_2 & \cdot \end{vmatrix} \quad i.e., \quad a_2b_3c_1 + a_3b_1c_2 = \mathrm{C}_1 \text{ say},$$

$$\begin{vmatrix} \cdot & a_2 & a_4 \\ b_1 & \cdot & b_4 \\ d_1 & d_2 & \cdot \end{vmatrix} \quad i.e., \quad a_2b_4d_1 + a_4b_1d_2 = \mathrm{C}_2 \text{ say},$$

$$\begin{vmatrix} \cdot & a_3 & a_4 \\ c_1 & \cdot & c_4 \\ d_1 & d_3 & \cdot \end{vmatrix} \quad i.e., \quad a_3c_4d_1 + a_4c_1d_3 = \mathrm{C}_3 \text{ say},$$

$$\begin{vmatrix} \cdot & b_3 & b_4 \\ c_2 & \cdot & c_4 \\ d_2 & d_3 & \cdot \end{vmatrix} \quad i.e., \quad b_3c_4d_2 + b_4c_2d_3 = \mathrm{C}_4 \text{ say},$$

$$\begin{vmatrix} \cdot & a_2 \\ b_1 & \cdot \end{vmatrix} \quad i.e., \; -a_2b_1 = \mathrm{B}_1 \text{ say,}$$

$$\begin{vmatrix} \cdot & a_3 \\ c_1 & \cdot \end{vmatrix} \quad i.e., \; -a_3c_1 = \mathrm{B}_2 \text{ say,}$$

$$\begin{vmatrix} \cdot & a_4 \\ d_1 & \cdot \end{vmatrix} \quad i.e., \; -a_4d_1 = \mathrm{B}_3 \text{ say,}$$

$$\begin{vmatrix} \cdot & b_3 \\ c_2 & \cdot \end{vmatrix} \quad i.e., \; -b_3c_2 = \mathrm{B}_4 \text{ say,}$$

$$\begin{vmatrix} \cdot & b_4 \\ d_2 & \cdot \end{vmatrix} \quad i.e., \; -b_4d_2 = \mathrm{B}_5 \text{ say,}$$

$$\begin{vmatrix} \cdot & c_4 \\ d_3 & \cdot \end{vmatrix} \quad i.e., \; -c_4d_3 = \mathrm{B}_6 \text{ say.}$$

Using the last six equations to eliminate b_1, c_1, d_1, c_2, d_2, d_3—these being the elements on one side of the primary diagonal of $|a_1b_2c_3d_4|$—from the preceding five equations, we have

$$\left.\begin{array}{r}
\mathrm{B}_1\mathrm{B}_6+\mathrm{B}_2\mathrm{B}_5+\mathrm{B}_3\mathrm{B}_4-\left[\begin{array}{l}
\quad \mathrm{B}_2\mathrm{B}_6\dfrac{a_2b_4}{a_3c_4} + \mathrm{B}_1\mathrm{B}_5\dfrac{a_3c_4}{a_2b_4} \\
- \; \mathrm{B}_3\dfrac{a_2b_3c_4}{a_4} - \mathrm{B}_1\mathrm{B}_4\mathrm{B}_6\dfrac{a_4}{a_2b_3c_4} \\
+\mathrm{B}_2\mathrm{B}_5\dfrac{a_4b_3}{a_3b_4} + \mathrm{B}_3\mathrm{B}_4\dfrac{a_3b_4}{a_4b_3}
\end{array}\right] = \mathrm{D}, \\
- \quad \mathrm{B}_2\dfrac{a_2b_3}{a_3} + \mathrm{B}_1\mathrm{B}_4\dfrac{a_3}{a_2b_3} \qquad = \mathrm{C}_1, \\
- \quad \mathrm{B}_3\dfrac{a_2b_4}{a_4} + \mathrm{B}_1\mathrm{B}_5\dfrac{a_4}{a_2b_4} \qquad = \mathrm{C}_2, \\
- \quad \mathrm{B}_3\dfrac{a_3c_4}{a_4} + \mathrm{B}_2\mathrm{B}_6\dfrac{a_4}{a_3c_4} \qquad = \mathrm{C}_3, \\
- \quad \mathrm{B}_5\dfrac{b_3c_4}{b_4} + \mathrm{B}_4\mathrm{B}_6\dfrac{b_4}{b_3c_4} \qquad = \mathrm{C}_4.
\end{array}\right\}$$

But the four fractional quantities $\dfrac{a_2b_3}{a_3}, \dfrac{a_2b_4}{a_4}, \dfrac{a_3c_4}{a_4}, \dfrac{b_3c_4}{b_4}$ — or say γ_1, γ_2, γ_3, γ_4 — in the last four equations are connected by the relation

$$\gamma_1\gamma_3 = \gamma_2\gamma_4 ,$$

and the three similar quantities in the remaining equation of the set are expressible in terms of these four, viz.:—

$$\frac{a_2b_4}{a_3c_4} = \frac{\gamma_2}{\gamma_3} \text{ or } \frac{\gamma_1}{\gamma_4},$$

$$\frac{a_2b_3c_4}{a_4} = \gamma_1\gamma_3 \text{ or } \gamma_2\gamma_4 ,$$

$$\frac{a_4b_3}{a_3b_4} = \frac{\gamma_1}{\gamma_2} \text{ or } \frac{\gamma_4}{\gamma_3}.$$

It is thus possible by the elimination of γ_1, γ_2, γ_3, γ_4 to deduce five equations, not more than two of which, however, can be independent.

5. Taking the first four equations of the set of five, and using γ_1, γ_2, γ_3 as just indicated, we have

$$\left.\begin{aligned}
B_1B_6+B_2B_5+B_3B_4-D &= \left(B_2B_6\frac{\gamma_2}{\gamma_3}+B_1B_5\frac{\gamma_3}{\gamma_2}\right)-\left(B_3\gamma_1\gamma_3+B_1B_4B_6\frac{1}{\gamma_1\gamma_3}\right)+\left(B_2B_5\frac{\gamma_1}{\gamma_2}+B_3B_4\frac{\gamma_2}{\gamma_1}\right)\\
C_1 &= -B_2\gamma_1+B_1B_4\frac{1}{\gamma_1},\\
C_2 &= -B_3\gamma_2+B_1B_5\frac{1}{\gamma_2},\\
C_3 &= -B_3\gamma_3+B_2B_6\frac{1}{\gamma_3}.
\end{aligned}\right\}$$

Now, by means of each pair of the last three of these equations, the γ's may be eliminated from a corresponding one of the bracketed expressions in the first equation, the results of this action in fact being

$$B_2B_6\frac{\gamma_2}{\gamma_3}+B_1B_5\frac{\gamma_3}{\gamma_2} = \frac{-C_2C_3+\sqrt{C_2^2+4B_1B_3B_5}\,\sqrt{C_3^2+4B_2B_3B_6}}{2B_3},$$

$$B_3\gamma_1\gamma_3+B_1B_4B_6\frac{1}{\gamma_1\gamma_3} = \frac{-C_1C_3+\sqrt{C_1^2+4B_1B_2B_4}\,\sqrt{C_3^2+4B_2B_3B_6}}{2B_2},$$

$$B_2B_5\frac{\gamma_1}{\gamma_2}+B_3B_4\frac{\gamma_2}{\gamma_1} = \frac{-C_1C_2+\sqrt{C_1^2+4B_1B_2B_4}\,\sqrt{C_2^2+4B_1B_3B_5}}{2B_1}.$$

We thus have

$$\begin{aligned}
D = B_1B_6+B_2B_5+B_3B_4+\frac{C_2C_3}{2B_3}+\frac{C_3C_1}{2B_2}+\frac{C_1C_2}{2B_1}\\
-\frac{1}{2B_3}\sqrt{C_2^2+4B_1B_3B_5}\,\sqrt{C_3^2+4B_2B_3B_6}+\frac{1}{2B_2}\sqrt{C_3^2+4B_2B_3B_6}\,\sqrt{C_1^2+4B_1B_2B}\\
-\frac{1}{2B_1}\sqrt{C_1^2+4B_1B_2B_4}\,\sqrt{C_2^2+4B_1B_3B_5},
\end{aligned}$$

—a relation among ten of the eleven devertebrated coaxial minors of $|a_1b_2c_3d_4|$. Then as for each of the ten there is an expression in terms of the vertebrate coaxial minors, and, in the case of one of them, viz., D, this expression involves the original determinant $|a_1b_2c_3d_4|$, it is clear that we may deduce from this the result foreshadowed by MacMahon, viz., an expression for $|a_1b_2c_3d_4|$ in terms of its coaxial minors.

Making the actual substitutions in places where subsequent simplification is readily possible,* we find

$$\begin{aligned}
|a_1b_2c_3d_4| = \Sigma a_1|b_2c_3d_4| + \Sigma|a_1b_2|\,|c_3d_4| - 2\Sigma a_1b_2|c_3d_4| + 6a_1b_2c_3d_4 + \frac{C_1C_2}{2B_1} + \frac{C_1C_3}{2B_2} + \frac{C_2C_3}{2B_3}\\
-\frac{1}{2B_1}\sqrt{C_1^2+4B_1B_2B_4}\,\sqrt{C_2^2+4B_1B_3B_5}+\frac{1}{2B_2}\sqrt{C_1^2+4B_1B_2B_4}\,\sqrt{C_3^2+4B_2B_3B_6}\\
-\frac{1}{2B_3}\sqrt{C_2^2+4B_1B_3B_5}\,\sqrt{C_3^2+4B_2B_3B_6},
\end{aligned}$$

* In the case of each expression under a root-sign a certain amount of simplification is also possible, *e.g.*, we find

$$\begin{aligned}
C_2^2 + 4B_1B_3B_5 = |a_1b_3d_4|^2 - \Sigma 2\,a_1b_3d_4|\,|b_3d_4|a_1 + 4|a_1b_3d_4|a_1b_3d_4 + 4|a_1b_3|\,|a_1d_4|\,|b_3d_4|\\
+ \Sigma a_1^2|b_3d_4|^2 - \Sigma 2a_1b_3|a_1d_4\ \ b_3d_4|.
\end{aligned}$$

where $B_1, B_2, \ldots B_6, C_1, C_2, C_3$ have the significations given to them in section 4, but are to be replaced by using the theorem of sections 2, 3.

6. Again, taking the last four of the set of five equations in section 4, and bearing in mind that $\gamma_1\gamma_3 = \gamma_2\gamma_4$, all that is necessary for elimination is to put

$$\gamma_1 = \frac{-C_1 + \sqrt{C_1^2+4B_1B_2B_4}}{2B_2} \quad \text{or} \quad \frac{2B_1B_4}{C_1 + \sqrt{C_1^2+4B_1B_2B_4}},$$
$$\gamma_2 = \frac{-C_2 + \sqrt{C_2^2+4B_1B_3B_5}}{2B_3} \quad \text{or} \quad \frac{2B_1B_5}{C_2 + \sqrt{C_2^2+4B_1B_3B_5}},$$
$$\gamma_3 = \frac{-C_3 + \sqrt{C_3^2+4B_2B_3B_6}}{2B_1} \quad \text{or} \quad \frac{2B_2B_6}{C_3 + \sqrt{C_3^2+4B_2B_3B_6}},$$

in the equation

$$C_4 = -B_5\frac{\gamma_1\gamma_3}{\gamma_2} + B_4B_6\frac{\gamma_2}{\gamma_1\gamma_3}.$$

The result of this action is

$$\begin{aligned} 4B_1B_2B_3C_4 + C_1C_2C_3 &- C_1\sqrt{C_2^2+4B_1B_3B_5}\sqrt{C_3^2+4B_2B_3B_6} \\ &+ C_2\sqrt{C_3^2+4B_2B_3B_6}\sqrt{C_1^2+4B_1B_2B_4} \\ &- C_3\sqrt{C_1^2+4B_1B_2B_4}\sqrt{C_2^2+4B_1B_3B_5} = 0\,; \end{aligned}$$

and similar equations can be got for C_3 in terms of C_1, C_2, C_4; for C_2 in terms of C_1, C_3, C_4; and for C_1 in terms of C_2, C_3, C_4.

7. On comparison of these results with those of Professor NANSON it will be found that instead of an explicit expression for $|a_1b_2c_3d_4|$ in terms of its coaxial minors, and an explicit expression for one of the coaxial minors of the 3rd order in terms of the three others and those of lower order, he obtains in each case an unsolved biquadratic equation. The presumption therefore is that each of his biquadratics must be resolvable into linear factors. This will now be shown to be the case. The series of necessary transformations is among the most interesting of the kind, and therefore well worthy of attention apart altogether from the problem with which they are here connected.

8. The latter of the two biquadratics is

$$\begin{vmatrix} DL & CQ & BR & AQRL+2BCD \\ CP & DM & AR & BRPM+2CAD \\ BP & AQ & DN & CPQN+2ABD \\ AL & BM & CN & DLMN+2ABC \end{vmatrix} = 0,$$

where

$$D, C, B, A\,;\ R, Q, L, P, M, N$$

correspond to but are not identical with the

$$C_1, C_2, C_3, C_4\,;\ B_1, B_2, B_3, B_4, B_5, B_6$$

of the present paper.

Now this determinant is easily seen to be the same as

$$\frac{1}{L^2M^2N^2}\begin{vmatrix} DLMN & CQMN & BRMN & AQRLMN+2BCDMN \\ CPNL & DMNL & ARNL & BRPMNL+2CADNL \\ BPLM & AQLM & DNLM & CPQNLM+2ABDLM \\ AL & BM & CN & DLMN+2ABC \end{vmatrix}.$$

Taking BC/L times each element of the last row from the corresponding element of the 1st row, CA/M times each element of the last row from the corresponding element of the 2nd row, and AB/N times each element of the last row from the corresponding element of the 3rd row, we transform this new determinant into

$$\begin{vmatrix} DLMN-ABC & CQMN-B^2C\frac{M}{L} & BRMN-BC^2\frac{N}{L} & AQRLMN+BCDMN-\frac{2AB^2C^2}{L} \\ CPNL-A^2C\frac{L}{M} & DLMN-ABC & ARNL-AC^2\frac{N}{M} & BRPLMN+CADLN-\frac{2A^2BC^2}{M} \\ BPLM-A^2B\frac{L}{M} & AQLM-AB^2\frac{M}{N} & DLMN-ABC & CPQLMN+ABDLM-\frac{2A^2B^2C}{N} \\ AL & BM & CN & DLMN+2ABC \end{vmatrix}.$$

Diminishing now each element of the last column by BC/L times the corresponding element of the 1st column, by CA/M times the corresponding element of the 2nd column, and by AB/N times the corresponding element of the 3rd column, we change the last column into

$$\left.\begin{matrix} AQRLMN-AC^2QN-AB^2RM+\frac{AB^2C^2}{L} \\ BRPLMN-BA^2RL-BC^2PN+\frac{A^2BC^2}{M} \\ CPQLMN-CB^2PM-CA^2QL+\frac{A^2B^2C}{N} \\ DLMN-ABC \end{matrix}\right\} \text{ or } \left\{\begin{matrix} \frac{A}{L}(NLQ-B^2)(LMR-C^2) \\ \frac{B}{M}(LMR-C^2)(MNP-A^2) \\ \frac{C}{N}(MNP-A^2)(NLQ-B^2) \\ DLMN-ABC \end{matrix}\right. ;$$

and if, merely for shortness' sake, we put

$$PL\left(1-\frac{A^2}{MNP}\right)=\lambda^2,$$

$$QM\left(1-\frac{B^2}{NLQ}\right)=\mu^2,$$

$$RN\left(1-\frac{C^2}{LMR}\right)=\nu^2,$$

the determinant becomes

$$\begin{vmatrix} DLMN-ABC & CN\mu^2 & BM\nu^2 & AL\mu^2\nu^2 \\ CN\lambda^2 & DLMN-ABC & AL\nu^2 & BM\nu^2\lambda^2 \\ BM\lambda^2 & AL\mu^2 & DLMN-ABC & CN\lambda^2\mu^2 \\ AL & BM & CN & DLMN-ABC \end{vmatrix}.$$

Dividing the columns by λ, μ, ν, $\lambda\mu\nu$ respectively, and multiplying the rows in order by the same, we obtain

$$\begin{vmatrix} \mathrm{DLMN}-\mathrm{ABC} & \mathrm{CN}\lambda\mu & \mathrm{BM}\nu\lambda & \mathrm{AL}\mu\nu \\ \mathrm{CN}\lambda\mu & \mathrm{DLMN}-\mathrm{ABC} & \mathrm{AL}\mu\nu & \mathrm{BM}\nu\lambda \\ \mathrm{BM}\nu\lambda & \mathrm{AL}\mu\nu & \mathrm{DLMN}-\mathrm{ABC} & \mathrm{CN}\lambda\mu \\ \mathrm{AL}\mu\nu & \mathrm{BM}\nu\lambda & \mathrm{CN}\lambda\mu & \mathrm{DLMN}-\mathrm{ABC} \end{vmatrix},$$

—a determinant which is seen to have all the elements of the primary diagonal alike, all the elements of the secondary diagonal alike, and to be symmetric with respect to both diagonals. Such a determinant, when of the 4th order, must clearly be a function of the four elements which necessarily recur in every line; and, as a matter of fact, it is known to be expressible as the product of four factors, the first of which is the sum of the said four elements, and differs from each of the others in the sign of two of its last three terms. The biquadratic we began with is thus the same as

$$\begin{aligned} &(\mathrm{DLMN}-\mathrm{ABC}+\mathrm{CN}\lambda\mu+\mathrm{BM}\nu\lambda+\mathrm{AL}\mu\nu) \\ .&(\mathrm{DLMN}-\mathrm{ABC}+\mathrm{CN}\lambda\mu-\mathrm{BM}\nu\lambda-\mathrm{AL}\mu\nu) \\ .&(\mathrm{DLMN}-\mathrm{ABC}-\mathrm{CN}\lambda\mu+\mathrm{BM}\nu\lambda-\mathrm{AL}\mu\nu) \\ .&(\mathrm{DLMN}-\mathrm{ABC}-\mathrm{CN}\lambda\mu-\mathrm{BM}\nu\lambda+\mathrm{AL}\mu\nu) = 0; \end{aligned}$$

so that if we put back the values of λ, μ, ν and solve, we have

$$\begin{aligned} \mathrm{D}=\frac{1}{\mathrm{LMN}}\{\mathrm{ABC} \pm \mathrm{C}\sqrt{\mathrm{A}^2-\mathrm{MNP}}\sqrt{\mathrm{B}^2-\mathrm{NLQ}} &\pm \mathrm{B}\sqrt{\mathrm{C}^2-\mathrm{LMR}}\sqrt{\mathrm{A}^2-\mathrm{MNP}} \\ &\pm \mathrm{C}\sqrt{\mathrm{A}^2-\mathrm{MNP}}\sqrt{\mathrm{B}^2-\mathrm{NLQ}}\}, \end{aligned}$$

and this, on the required changes being made, will be found to be identical with the result of section 6.

9. The other biquadratic referred to is

$$\begin{vmatrix} \theta & (1-\mathrm{C})\sqrt{1-\mathrm{B}^2} & (1-\mathrm{B})\sqrt{1-\mathrm{C}^2} & (1-\mathrm{A})\sqrt{1-\mathrm{B}^2}\sqrt{1-\mathrm{C}^2} \\ (1-\mathrm{C})\sqrt{1-\mathrm{A}^2} & \theta & (1-\mathrm{A})\sqrt{1-\mathrm{C}^2} & (1-\mathrm{B})\sqrt{1-\mathrm{C}^2}\sqrt{1-\mathrm{A}^2} \\ (1-\mathrm{B})\sqrt{1-\mathrm{A}^2} & (1-\mathrm{A})\sqrt{1-\mathrm{B}^2} & \theta & (1-\mathrm{C})\sqrt{1-\mathrm{A}^2}\sqrt{1-\mathrm{B}^2} \\ 1-\mathrm{A} & 1-\mathrm{B} & 1-\mathrm{C} & \theta \end{vmatrix} = 0,$$

where θ stands for $(\mathrm{A}-1)(\mathrm{B}-1)(\mathrm{C}-1)-\frac{1}{2}\Delta$. It is the biquadratic not for the general determinant $|a_1b_2c_3d_4|$ but for the very special instance

$$\begin{vmatrix} 1 & h & g^{-1} & 1 \\ h^{-1} & 1 & f & 1 \\ g & f^{-1} & 1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} \quad \text{or} \quad \Delta.$$

In this case the required transformation is very easy. All that is necessary is to divide the first three rows by $\sqrt{1-\mathrm{B}^2}\sqrt{1-\mathrm{C}^2}$, $\sqrt{1-\mathrm{C}^2}\sqrt{1-\mathrm{A}^2}$, $\sqrt{1-\mathrm{A}^2}\sqrt{1-\mathrm{B}^2}$ respectively, and then multiply in order the first three columns by the same. The result is

$$\begin{vmatrix} \theta & (1-C)\sqrt{1-A^2}\sqrt{1-B^2} & (1-B)\sqrt{1-C^2}\sqrt{1-A^2} & (1-A)\sqrt{1-B^2}\sqrt{1-C^2} \\ (1-C)\sqrt{1-A^2}\sqrt{1-B^2} & \theta & (1-A)\sqrt{1-B^2}\sqrt{1-C^2} & (1-B)\sqrt{1-C^2}\sqrt{1-A^2} \\ (1-B)\sqrt{1-C^2}\sqrt{1-A^2} & (1-A)\sqrt{1-B^2}\sqrt{1-C^2} & \theta & (1-C)\sqrt{1-A^2}\sqrt{1-B^2} \\ (1-A)\sqrt{1-B^2}\sqrt{1-C^2} & (1-B)\sqrt{1-C^2}\sqrt{1-A^2} & (1-C)\sqrt{1-A^2}\sqrt{1-B^2} & \theta \end{vmatrix} = 0,$$

where again the determinant has the elements of the primary diagonal all alike, the elements of the secondary diagonal all alike, and is symmetric with respect to both diagonals. As before, therefore, it resolves into four factors, and we have on substituting the value of θ

$$(A-1)(B-1)(C-1) - \tfrac{1}{2}\Delta \pm (1-C)\sqrt{1-A^2}\sqrt{1-B^2} \pm (1-B)\sqrt{1-C^2}\sqrt{1-A^2} \\ \pm (1-A)\sqrt{1-A^2}\sqrt{1-B^2} = 0,$$

or

$$\Delta = 2ABC - 2\Sigma AB + 2\Sigma A - 2 \pm 2(1-C)\sqrt{1-A^2}\sqrt{1-B^2} \pm 2(1-B)\sqrt{1-B^2}\sqrt{1-A^2} \\ \pm 2(1-A)\sqrt{1-A^2}\sqrt{1-B^2},$$

which is readily shown to be in agreement with the more general result in section 5.*

10. Not only does the determinant

$$\begin{vmatrix} DL & CQ & BR & AQRL+2BCD \\ CP & DM & AR & BRPM+2CAD \\ BP & AQ & DN & CPQN+2ABD \\ AL & BM & CN & DLMN+2ABC \end{vmatrix}$$

resolve into factors, but each of the two determinants into which it may be partitioned is also so resolvable. For, multiplying the columns in order by $\sqrt{MNQR}$, $\sqrt{NLRP}$, $\sqrt{LMPQ}$, 1, and then dividing the rows in order by $\sqrt{LQR}$, $\sqrt{MRP}$, $\sqrt{NPQ}$, $\sqrt{LMN}$, we obtain the new form

$$\begin{vmatrix} D\sqrt{LMN} & C\sqrt{NPQ} & B\sqrt{MRP} & A\sqrt{LQR} + \dfrac{2BCD}{\sqrt{LQR}} \\ C\sqrt{NPQ} & D\sqrt{LMN} & A\sqrt{LQR} & B\sqrt{MRP} + \dfrac{2CDA}{\sqrt{MRP}} \\ B\sqrt{MRP} & A\sqrt{LQR} & D\sqrt{LMN} & C\sqrt{NPQ} + \dfrac{2DAB}{\sqrt{NPQ}} \\ A\sqrt{LQR} & B\sqrt{MRP} & C\sqrt{NPQ} & D\sqrt{LMN} + \dfrac{2ABC}{\sqrt{LMN}} \end{vmatrix};$$

* Instead of the biquadratic of this section another might readily have been obtained from the single equation numbered (9) in Professor NANSON's paper, viz.,

$$\mu + \sqrt{1-B^2}\sqrt{1-C^2} + \sqrt{1-C^2}\sqrt{1-A^2} + \sqrt{1-A^2}\sqrt{1-B^2} = 0,$$

where

$$\mu = A+B+C+D-\tfrac{1}{2}\Delta-1-BC-CA-AB.$$

Observe also that this equation gives a much simpler expression for Δ, viz.:—

$$\Delta = -2+2\Sigma A-2\Sigma AB+2\Sigma\sqrt{1-B^2}\sqrt{1-C^2}.$$

and on partitioning this into two the first is seen to be

$$\begin{aligned}
&= (\mathrm{D}\sqrt{\mathrm{LMN}} + \mathrm{C}\sqrt{\mathrm{NPQ}} + \mathrm{B}\sqrt{\mathrm{MRP}} + \mathrm{A}\sqrt{\mathrm{LQR}}) \\
&\quad .(\mathrm{D}\sqrt{\mathrm{LMN}} + \mathrm{C}\sqrt{\mathrm{NPQ}} - \mathrm{B}\sqrt{\mathrm{MRP}} - \mathrm{A}\sqrt{\mathrm{LQR}}) \\
&\quad .(\mathrm{D}\sqrt{\mathrm{LMN}} - \mathrm{C}\sqrt{\mathrm{NPQ}} + \mathrm{B}\sqrt{\mathrm{MRP}} - \mathrm{A}\sqrt{\mathrm{LQR}}) \\
&\quad .(\mathrm{D}\sqrt{\mathrm{LMN}} - \mathrm{C}\sqrt{\mathrm{NPQ}} - \mathrm{B}\sqrt{\mathrm{MRP}} + \mathrm{A}\sqrt{\mathrm{LQR}}),
\end{aligned}$$

and the second to be

$$\begin{vmatrix}
\mathrm{D}\sqrt{\mathrm{LMN}} & \mathrm{C}\sqrt{\mathrm{NPQ}} & \mathrm{B}\sqrt{\mathrm{MRP}} & \mathrm{D}\sqrt{\mathrm{LMN}}\,.\,\mathrm{C}\sqrt{\mathrm{NPQ}}\,.\,\mathrm{B}\sqrt{\mathrm{MRP}} \\
\mathrm{C}\sqrt{\mathrm{NPQ}} & \mathrm{D}\sqrt{\mathrm{LMN}} & \mathrm{A}\sqrt{\mathrm{LQR}} & \mathrm{C}\sqrt{\mathrm{NPQ}}\,.\,\mathrm{D}\sqrt{\mathrm{LMN}}\,.\,\mathrm{A}\sqrt{\mathrm{LQR}} \\
\mathrm{B}\sqrt{\mathrm{MRP}} & \mathrm{A}\sqrt{\mathrm{LQR}} & \mathrm{D}\sqrt{\mathrm{LMN}} & \mathrm{B}\sqrt{\mathrm{MRP}}\,.\,\mathrm{A}\sqrt{\mathrm{LQR}}\,.\,\mathrm{D}\sqrt{\mathrm{LMN}} \\
\mathrm{A}\sqrt{\mathrm{LQR}} & \mathrm{B}\sqrt{\mathrm{MRP}} & \mathrm{C}\sqrt{\mathrm{NPQ}} & \mathrm{A}\sqrt{\mathrm{LQR}}\,.\,\mathrm{B}\sqrt{\mathrm{MRP}}\,.\,\mathrm{C}\sqrt{\mathrm{NPQ}}
\end{vmatrix} \times \frac{2}{\mathrm{LMNPQR}},$$

and therefore

$$\begin{aligned}
&= (\mathrm{C}\sqrt{\mathrm{NPQ}}\,.\,\mathrm{A}\sqrt{\mathrm{LQR}} - \mathrm{D}\sqrt{\mathrm{LMN}}\,.\,\mathrm{B}\sqrt{\mathrm{MRP}}) \\
&\quad (\mathrm{A}\sqrt{\mathrm{LQR}}\,.\,\mathrm{B}\sqrt{\mathrm{MRP}} - \mathrm{C}\sqrt{\mathrm{NPQ}}\,.\,\mathrm{D}\sqrt{\mathrm{LMN}}) \\
&\quad (\mathrm{B}\sqrt{\mathrm{MRP}}\,.\,\mathrm{C}\sqrt{\mathrm{NPQ}} - \mathrm{A}\sqrt{\mathrm{LQR}}\,.\,\mathrm{D}\sqrt{\mathrm{LMN}}) \div \tfrac{1}{2}\mathrm{LMNPQR}, \\
&= 2(\mathrm{CAQ} - \mathrm{DBM})(\mathrm{ABR} - \mathrm{CDN})(\mathrm{BCP} - \mathrm{ADL}).
\end{aligned}$$

11. Were it not for the divisor LMNPQR attached to the second determinant in the preceding section, the full determinant would be a function of only four variables, viz. :—

$$\begin{gathered}
\mathrm{A}\sqrt{\mathrm{LQR}}, \\
\mathrm{B}\sqrt{\mathrm{MRP}}, \\
\mathrm{C}\sqrt{\mathrm{NPQ}}, \\
\mathrm{D}\sqrt{\mathrm{LMN}};
\end{gathered}$$

and as a matter of fact the final expansion of it may be written

$$\begin{aligned}
&\Sigma(\mathrm{A}\sqrt{\mathrm{LQR}})^4 - 2\Sigma(\mathrm{A}\sqrt{\mathrm{LQR}})^2(\mathrm{B}\sqrt{\mathrm{MRP}})^2 \\
&\qquad + 8(\mathrm{A}\sqrt{\mathrm{LQR}}\,.\,\mathrm{B}\sqrt{\mathrm{MRP}}\,.\,\mathrm{C}\sqrt{\mathrm{NPQ}}\,.\,\mathrm{D}\sqrt{\mathrm{LMN}}) \\
&\qquad + \frac{4\Sigma(\mathrm{A}\sqrt{\mathrm{LQR}})^2(\mathrm{B}\sqrt{\mathrm{MRP}})^2(\mathrm{C}\sqrt{\mathrm{NPQ}})^2 - 4\Sigma(\mathrm{A}\sqrt{\mathrm{LQR}})^3.\mathrm{B}\sqrt{\mathrm{MRP}}.\mathrm{C}\sqrt{\mathrm{NPQ}}.\mathrm{D}\sqrt{\mathrm{LMN}}}{\mathrm{LMNPQR}}.
\end{aligned}$$

12. Standing in close connection with the subject-matter of the preceding sections—the connection of general with particular—is the problem of clearing the equation

$$x + h\sqrt{bc} + k\sqrt{ca} + l\sqrt{ab} = 0$$

of root-signs, or of transforming a fraction of which $x + h\sqrt{bc} + k\sqrt{ca} + l\sqrt{ab}$ is the denominator into one having its denominator rational. Viewing the matter in either way we reach the result

$$(x+h\sqrt{bc}+k\sqrt{ca}+l\sqrt{ab})(x+h\sqrt{bc}-k\sqrt{ca}-l\sqrt{ab})(x-h\sqrt{bc}+k\sqrt{ca}-l\sqrt{ab})(x-h\sqrt{bc}-k\sqrt{ca}+l\sqrt{ab})$$

or

$$\begin{aligned}
&x^4 + h^4b^2c^2 + k^4c^2a^2 + l^4a^2b^2 \\
&\quad - 2x^2(h^2bc + k^2ca + l^2ab) \\
&\quad - 2abc(h^2k^2c + k^2l^2a + l^2h^2b) \\
&\quad - 8xhklabc.
\end{aligned}$$

This, however, is well known to be equal to the determinant

$$\begin{vmatrix} x & h\sqrt{bc} & k\sqrt{ca} & l\sqrt{ab} \\ h\sqrt{bc} & x & l\sqrt{ab} & k\sqrt{ca} \\ k\sqrt{ca} & l\sqrt{ab} & x & h\sqrt{bc} \\ l\sqrt{ab} & k\sqrt{ca} & h\sqrt{bc} & x \end{vmatrix};$$

and we may consequently say that *the rationalizant of the expression* $x+h\sqrt{bc}+k\sqrt{ca}+l\sqrt{ab}$ *is the biaxisymmetric determinant of the 4th order which has the terms of the expression for the elements of its first row, all the elements of its primary diagonal alike, and all the elements of its secondary diagonal alike.*

Another determinant form of the result is obtained by using the dialytic method of elimination. Taking the original equation and multiplying in succession by $\sqrt{bc}$, $\sqrt{ca}$, $\sqrt{ab}$, we have

$$\left.\begin{array}{l} x + h\sqrt{bc} + k\sqrt{ca} + l\sqrt{ab} = 0 \\ hbc + x\sqrt{bc} + lb\sqrt{ca} + kc\sqrt{ab} = 0 \\ kca + la\sqrt{bc} + x\sqrt{ca} + hc\sqrt{ab} = 0 \\ lab + ka\sqrt{bc} + hb\sqrt{ca} + x\sqrt{ab} = 0 \end{array}\right\},$$

and therefore on eliminating $\sqrt{bc}$, $\sqrt{ca}$, $\sqrt{ab}$ there results the rationalizant

$$\begin{vmatrix} x & h & k & l \\ hbc & x & lb & kc \\ kca & la & x & hc \\ lab & ka & hb & x \end{vmatrix}.$$

It is easy to change the one form into the other; indeed, this change is what has been effected in sections 8, 9, Professor Nanson having obtained his results in the latter of the two forms.

13. Another closely related problem, as Professor Nanson has made clear, is that of expressing $\cos(\alpha+\beta+\gamma)$ in terms of $\cos\alpha$, $\cos\beta$, $\cos\gamma$, or say, for shortness' sake, S in terms of A, B, C.

Since

$$\cos(\alpha+\beta+\gamma) - \cos\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\sin\gamma + \cos\beta\sin\gamma\sin\alpha + \cos\gamma\sin\alpha\sin\beta = 0,$$

we have

$$S - ABC + A\sqrt{1-B^2}\sqrt{1-C^2} + B\sqrt{1-C^2}\sqrt{1-A^2} + C\sqrt{1-A^2}\sqrt{1-B^2} = 0,$$

and the problem is seen to be a case of the preceding, the result being either

$$\begin{vmatrix} S-ABC & A\sqrt{1-B^2}\sqrt{1-C^2} & B\sqrt{1-C^2}\sqrt{1-A^2} & C\sqrt{1-A^2}\sqrt{1-B^2} \\ A\sqrt{1-B^2}\sqrt{1-C^2} & S-ABC & C\sqrt{1-A^2}\sqrt{1-B^2} & B\sqrt{1-C^2}\sqrt{1-A^2} \\ B\sqrt{1-C^2}\sqrt{1-A^2} & C\sqrt{1-A^2}\sqrt{1-B^2} & S-ABC & A\sqrt{1-B^2}\sqrt{1-C^2} \\ C\sqrt{1-A^2}\sqrt{1-B^2} & B\sqrt{1-C^2}\sqrt{1-A^2} & A\sqrt{1-B^2}\sqrt{1-C^2} & S-ABC \end{vmatrix}$$

or

$$\begin{vmatrix} S-ABC & A & B & C \\ A(1-B^2)(1-C^2) & S-ABC & C(1-B^2) & B(1-C^2) \\ B(1-C^2)(1-A^2) & C(1-A^2) & S-ABC & A(1-C^2) \\ C(1-A^2)(1-B^2) & B(1-A^2) & A(1-B^2) & S-ABC \end{vmatrix},$$

which latter can be simplified, as Professor NANSON shows, into

$$\begin{vmatrix} S+2ABC & A & B & C \\ A+2BCS & S & C & B \\ B+2ACS & C & S & A \\ C+2ABS & B & A & S \end{vmatrix}.$$

This, however, can be obtained much more directly from the use of another expression for $\cos(\alpha+\beta+\gamma)$, viz. :—

$$\cos(\alpha+\beta+\gamma) = \cos\alpha\cos(\beta+\gamma) + \cos\beta\cos(\gamma+\alpha) + \cos\gamma\cos(\alpha+\beta) - 2\cos\alpha\cos\beta\cos\gamma,$$

where nothing but cosines appears, the angles being

$$\alpha,\ \beta,\ \gamma;\ \ \beta+\gamma,\ \gamma+\alpha,\ \alpha+\beta;\ \ \alpha+\beta+\gamma.$$

Making in this equation the substitutions

$$\begin{cases} \alpha = \alpha+\beta+\gamma, \\ \beta = -\gamma, \\ \gamma = -\beta, \end{cases} \quad \begin{cases} \alpha = -\gamma, \\ \beta = \alpha+\beta+\gamma, \\ \gamma = -\alpha, \end{cases} \quad \begin{cases} \alpha = -\beta, \\ \beta = -\alpha, \\ \gamma = \alpha+\beta+\gamma, \end{cases}$$

we obtain three other perfectly similar identities* connecting the same seven cosines, the complete set of four identities being in the notation above employed

$$\left.\begin{aligned} S + 2ABC - A\cos(\beta+\gamma) - B\cos(\gamma+\alpha) - C\cos(\alpha+\beta) &= 0 \\ A + 2SCB - S\cos(\beta+\gamma) - C\cos(\gamma+\alpha) - B\cos(\alpha+\beta) &= 0 \\ B + 2CSA - C\cos(\beta+\gamma) - S\cos(\gamma+\alpha) - A\cos(\alpha+\beta) &= 0 \\ C + 2BAS - B\cos(\beta+\gamma) - A\cos(\gamma+\alpha) - S\cos(\alpha+\beta) &= 0 \end{aligned}\right\}.$$

From these $\cos(\beta+\gamma)$, $\cos(\gamma+\alpha)$, $\cos(\alpha+\beta)$ can be eliminated, and the desired result at once obtained.

14. It may be noticed in passing that the substitution of $90°-\alpha$, $90°-\beta$, $90°-\gamma$ for α, β, γ gives the similar relation between $\sin(\alpha+\beta+\gamma)$, $\sin\alpha$, $\sin\beta$, $\sin\gamma$.

It should also be noted that the corresponding expression for $\cos(\alpha+\beta)$ in terms of $\cos\alpha$ and $\cos\beta$ is obtained from an identity of a different type, viz., $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, the set of equations being

$$\left.\begin{aligned} \sin\beta + \cos(\alpha+\beta).\sin\alpha - \cos\alpha\sin(\alpha+\beta) &= 0 \\ \cos(\alpha+\beta).\sin\beta + \sin\alpha - \cos\beta\sin(\alpha+\beta) &= 0 \\ \cos\alpha.\sin\beta + \cos\beta.\sin\alpha - \sin(\alpha+\beta) &= 0 \end{aligned}\right\}$$

* In effect the substitutions are the same as the circular substitution $\begin{pmatrix} S & A & B & C \\ A & B & C & S \end{pmatrix}$ if we consider $\cos(\beta+\gamma)$ $\cos(\gamma+\alpha)$, $\cos(\alpha+\beta)$ as invariant.

and the resulting equation *

$$\begin{vmatrix} 1 & \cos(\alpha+\beta) & \cos\alpha \\ \cos(\alpha+\beta) & 1 & \cos\beta \\ \cos\alpha & \cos\beta & 1 \end{vmatrix} = 0 .$$

The same identity almost suffices to give the corresponding relation between $\sin(\alpha+\beta)$, $\sin\alpha$, $\sin\beta$, the set of equations now being

$$\left.\begin{array}{llllll} & -\sin\beta\,.\cos\alpha & -\sin\alpha\,.\cos\beta & + & \sin(\alpha+\beta) & = 0 \\ \sin\beta\,.\cos(\alpha+\beta) & & -\sin(\alpha+\beta)\,.\cos\beta & + & \sin\alpha & = 0 \\ \sin\alpha\,.\cos(\alpha+\beta) & -\sin(\alpha+\beta)\,.\cos\alpha & & + & \sin\beta & = 0 \\ \sin(\alpha+\beta)\,.\cos(\alpha+\beta) & -\sin\alpha\,.\cos\alpha & -\sin\beta\,.\cos\beta & + & 2\sin\alpha\sin\beta\sin(\alpha+\beta) & = 0 \end{array}\right\} ,$$

whence on the elimination of $\cos\alpha$, $\cos\beta$, $\cos(\alpha+\beta)$ we have

$$\begin{vmatrix} . & \sin\beta & \sin\alpha & \sin(\alpha+\beta) \\ \sin\beta & . & \sin(\alpha+\beta) & \sin\alpha \\ \sin\alpha & \sin(\alpha+\beta) & . & \sin\beta \\ \sin(\alpha+\beta) & \sin\alpha & \sin\beta & 2\sin\alpha\sin\beta\sin(\alpha+\beta) \end{vmatrix} = 0 .$$

15. The consideration of the relation between $\cos(\alpha+\beta+\gamma+\delta)$ and $\cos\alpha$, $\cos\beta$, $\cos\gamma$, $\cos\delta$ leads at once to the question of the rationalization of the equation

$$a + b\sqrt{xy} + c\sqrt{xz} + d\sqrt{xw} + e\sqrt{yz} + f\sqrt{yw} + g\sqrt{zw} + h\sqrt{xyzw} = 0,$$

because

$$\cos(\alpha+\beta+\gamma+\delta) = \cos\alpha\cos\beta\cos\gamma\cos\delta - \Sigma\cos\gamma\cos\delta\sin\alpha\sin\beta + \sin\alpha\sin\beta\sin\gamma\sin\delta .$$

By proceeding in exactly the same manner as in section 12 the result of the rationalization is obtained in three forms, viz., (1) the product

$$\begin{array}{l} (a + b\sqrt{xy} + c\sqrt{xz} + d\sqrt{xw} + e\sqrt{yz} + f\sqrt{yw} + g\sqrt{zw} + h\sqrt{xyzw}) \\ .(a + b\sqrt{xy} + c\sqrt{xz} + d\sqrt{xw} - e\sqrt{yz} - f\sqrt{yw} - g\sqrt{zw} - h\sqrt{xyzw}) \\ .(a + b\sqrt{xy} - c\sqrt{xz} - d\sqrt{xw} + e\sqrt{yz} + f\sqrt{yw} - g\sqrt{zw} - h\sqrt{xyzw}) \\ .(a + b\sqrt{xy} - c\sqrt{xz} - d\sqrt{xw} - e\sqrt{yz} - f\sqrt{yw} + g\sqrt{zw} + h\sqrt{xyzw}) \\ .(a - b\sqrt{xy} + c\sqrt{xz} - d\sqrt{xw} + e\sqrt{yz} - f\sqrt{yw} + g\sqrt{zw} - h\sqrt{xyzw}) \\ .(a - b\sqrt{xy} + c\sqrt{xz} - d\sqrt{xw} - e\sqrt{yz} + f\sqrt{yw} - g\sqrt{zw} + h\sqrt{xyzw}) \\ .(a - b\sqrt{xy} - c\sqrt{xz} + d\sqrt{xw} + e\sqrt{yz} - f\sqrt{yw} - g\sqrt{zw} + h\sqrt{xyzw}) \\ .(a - b\sqrt{xy} - c\sqrt{xz} + d\sqrt{xw} - e\sqrt{yz} + f\sqrt{yw} + g\sqrt{zw} - h\sqrt{xyzw}) , \end{array}$$

* It is interesting to note the mode in which the more general relation connecting $\cos(\alpha+\beta+\gamma)$, $\cos\alpha$, $\cos\beta$, $\cos\gamma$, passes over into this on putting $\gamma=0$ in the former. The result of the substitution is

$$\begin{vmatrix} \cos\alpha & 1 & \cos(\alpha+\beta) & \cos\beta+2\cos\alpha\cos(\alpha+\beta) \\ \cos\beta & \cos(\alpha+\beta) & 1 & \cos\alpha+2\cos\beta\cos(\alpha+\beta) \\ 1 & \cos\alpha & \cos\beta & \cos(\alpha+\beta)+2\cos\alpha\cos\beta \\ \cos(\alpha+\beta) & \cos\beta & \cos\alpha & 1+2\cos\alpha\cos\beta\cos(\alpha+\beta) \end{vmatrix} ,$$

where the elements of the 4th column are easily transformed into zeros with the exception of the last element which becomes

$$1 + 2\cos\alpha\cos\beta\cos(\alpha+\beta) - \cos^2(\alpha+\beta) - \cos^2\beta - \cos^2\alpha,$$

so that the value of the determinant is seen to be

$$\begin{vmatrix} \cos\alpha & 1 & \cos(\alpha+\beta) \\ \cos\beta & \cos(\alpha+\beta) & 1 \\ 1 & \cos\alpha & \cos\beta \end{vmatrix}^2 .$$

With this mode of degeneration may be compared that seen on p. 377 of *Proc. Roy. Soc. Edin.*, xx

(2) the biaxisymmetric determinant

$$\begin{vmatrix} a & b\sqrt{xy} & c\sqrt{xz} & d\sqrt{xw} & e\sqrt{yz} & f\sqrt{yw} & g\sqrt{zw} & h\sqrt{xyzw} \\ b\sqrt{xy} & a & e\sqrt{yz} & f\sqrt{yw} & c\sqrt{xz} & d\sqrt{xw} & h\sqrt{xyzw} & g\sqrt{zw} \\ c\sqrt{xz} & e\sqrt{yz} & a & g\sqrt{zw} & b\sqrt{xy} & h\sqrt{xyzw} & d\sqrt{xw} & f\sqrt{yw} \\ d\sqrt{xw} & f\sqrt{yw} & g\sqrt{zw} & a & h\sqrt{xyzw} & b\sqrt{xy} & c\sqrt{xz} & e\sqrt{yz} \\ e\sqrt{yz} & c\sqrt{xz} & b\sqrt{xy} & h\sqrt{xyzw} & a & g\sqrt{zw} & f\sqrt{yw} & d\sqrt{xw} \\ f\sqrt{yw} & d\sqrt{xw} & h\sqrt{xyzw} & b\sqrt{xy} & g\sqrt{zw} & a & e\sqrt{yz} & c\sqrt{xz} \\ g\sqrt{zw} & h\sqrt{xyzw} & d\sqrt{xw} & c\sqrt{xz} & f\sqrt{yw} & e\sqrt{yz} & a & b\sqrt{xy} \\ h\sqrt{xyzw} & g\sqrt{zw} & f\sqrt{yw} & e\sqrt{yz} & d\sqrt{xw} & c\sqrt{xz} & b\sqrt{xy} & a \end{vmatrix},$$

and (3) the axisymmetric determinant

$$\begin{vmatrix} a & b & c & d & e & f & g & h \\ bxy & a & ey & fy & cx & dx & hxy & g \\ cxz & ez & a & gz & bx & hxz & dx & f \\ dxw & fw & gw & a & hxw & bx & cx & e \\ eyz & cz & by & hyz & a & gz & fy & d \\ fyw & dw & hyw & by & gw & a & ey & c \\ gzw & hzw & dw & cz & fw & ez & a & b \\ hxyzw & gzw & fyw & eyz & dxw & cxz & bxy & a \end{vmatrix}.$$

The third form is easily changed into the second by multiplying the columns in order by

$$1,\ \sqrt{xy},\ \sqrt{xz},\ \sqrt{xw},\ \sqrt{yz},\ .\ .\ .\ .\ ,\ \sqrt{xyzw},$$

and then dividing the rows in order by the same. The mode of resolution of the second form into factors is well known.*

16. There is still another variant of the problem of sections 6, 8, viz., to express the relation

$$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z + \cos^{-1}w = 0$$

in purely algebraical form. In essence it is the same as the variant dealt with in section 13.

Subtracting $\cos^{-1}w$ from both sides, and then taking the cosines of the two equals, we have

$$xyz - x\sqrt{1-y^2}\sqrt{1-z^2} - y\sqrt{1-z^2}\sqrt{1-x^2} - z\sqrt{1-x^2}\sqrt{1-y^2} = w,$$

which is at once seen to be an equation of the form dealt with in section 12. The result of the rationalization is

$$\begin{vmatrix} x & y & z & w+2xyz \\ y & x & w & z+2yxw \\ z & w & x & y+2zwx \\ w & z & y & x+2wzy \end{vmatrix} = 0,$$

or

$$\Sigma x^4 - 2\Sigma x^2y^2 + 8xyzw + 4\Sigma x^2y^2z^2 - 4\,\Sigma\, x^3yzw = 0.$$

* See *Quart. Journ. of Math.*, xviii. pp. 170, 171.

Similarly we have for the equation

$$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 0$$

the purely algebraical equivalent

$$\begin{vmatrix} 1 & x & z \\ x & 1 & y \\ z & y & 1 \end{vmatrix} = 0;$$

and for the equation

$$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z + \cos^{-1}w + \cos^{-1}v = 0$$

a purely algebraical equivalent essentially the same as that referred to in section 15 as giving the relation between $\cos(\alpha+\beta+\gamma+\delta)$ and $\cos\alpha$, $\cos\beta$, $\cos\gamma$, $\cos\delta$.

17. This suggests a very simple and perfectly symmetrical mode of expressing the relation of section 6 between the coaxial minors of an order lower than the fourth, viz.:—

$$\cos^{-1}\frac{C_1}{2\sqrt{-B_1B_2B_4}} + \cos^{-1}\frac{C_2}{2\sqrt{-B_1B_3B_5}} + \cos^{-1}\frac{C_3}{2\sqrt{-B_2B_3B_6}} + \frac{C_4}{2\sqrt{-B_4B_5B_6}} = 0.$$

The law of formation of the denominators is perhaps not clear, but this is due merely to a defect in the notation. If we substitute for B's and C's their values as given in terms of the coaxial minors of $|a_1b_2c_3d_4|$ we have

$$\sum \cos^{-1}\frac{|a_1b_2c_3| - a_1|b_2c_3| - b_2|a_1c_3| - c_3|a_1b_2| + 2a_1b_2c_3}{2(-1)^{\frac{1}{2}}(|a_1b_2| - a_1b_2)^{\frac{1}{2}}(|a_1c_3| - a_1c_3)^{\frac{1}{2}}(|b_2c_3| - b_2c_3)^{\frac{1}{2}}} = 0;$$

and, further, if we denote by

$$\left|\begin{matrix} a_1b_2c_3 \\ 000 \end{matrix}\right|$$

the determinant got from $|a_1b_2c_3|$ by changing the elements of the primary diagonal into zeros, the relation may be written

$$\sum \cos^{-1}\frac{\begin{matrix} a_1b_2c_3 \\ 000 \end{matrix}}{2\left\{-\begin{matrix} a_1b_2 \\ 00 \end{matrix}\begin{matrix} a_1c_3 \\ 00 \end{matrix}\begin{matrix} b_2c_3 \\ 00 \end{matrix}\right\}^{\frac{1}{2}}} = 0.$$

18. Another matter which has light thrown upon it by certain of the preceding paragraphs is SYLVESTER's original illustration of the dialytic method of elimination as applied to ternary quadrics. It will be remembered that from the equations

$$\left.\begin{aligned} Bx^2 - 2C'xy + Ay^2 &= 0 \\ Cy^2 - 2A'yz + Bz^2 &= 0 \\ Az^2 - 2B'zx + Cx^2 &= 0 \end{aligned}\right\}$$

he deduced three others

$$\left.\begin{aligned} C'z^2 + Cxy - A'zx - B'yz &= 0 \\ A'x^2 + Ayz - B'xy - C'zx &= 0 \\ B'y^2 + Bzx - C'yz - A'xy &= 0 \end{aligned}\right\},$$

and thus obtained the eliminant in the form

$$\begin{vmatrix} \cdot & C & B & -2A' & \cdot & \cdot \\ C & \cdot & A & \cdot & -2B' & \cdot \\ B & A & \cdot & \cdot & \cdot & -2C' \\ A' & \cdot & \cdot & A & -C' & -B' \\ \cdot & B' & \cdot & -C' & B & -A' \\ \cdot & \cdot & C' & -B' & -A' & C \end{vmatrix},$$

which, it was afterwards shown,

$$= 2\begin{vmatrix} A & C' & B' \\ C' & B & A' \\ B' & A' & C \end{vmatrix}^2 .$$

Now the given equations may be written

$$\left.\begin{aligned} \frac{y\sqrt{A}}{x\sqrt{B}} + \frac{x\sqrt{B}}{y\sqrt{A}} &= \frac{2C'}{\sqrt{AB}} \\ \frac{z\sqrt{B}}{y\sqrt{C}} + \frac{y\sqrt{C}}{z\sqrt{B}} &= \frac{2A'}{\sqrt{BC}} \\ \frac{x\sqrt{C}}{z\sqrt{A}} + \frac{z\sqrt{A}}{x\sqrt{C}} &= \frac{2B'}{\sqrt{CA}} \end{aligned}\right\} ;$$

consequently it is seen that there exists the relation

$$\cos^{-1}\frac{A'}{\sqrt{BC}} + \cos^{-1}\frac{B'}{\sqrt{CA}} + \cos^{-1}\frac{C'}{\sqrt{AB}} = 0,$$

and therefore

$$\begin{vmatrix} 1 & \frac{A'}{\sqrt{BC}} & \frac{C'}{\sqrt{AB}} \\ \frac{A'}{\sqrt{BC}} & 1 & \frac{B'}{\sqrt{CA}} \\ \frac{C'}{\sqrt{AB}} & \frac{B'}{\sqrt{CA}} & 1 \end{vmatrix} = 0 .$$

Similarly the resultant of

$$\left.\begin{aligned} Bx^2 - Dxy + Ay^2 &= 0 \\ Cy^2 - Eyz + Bz^2 &= 0 \\ Lz^2 - Kzw + Cw^2 &= 0 \\ Aw^2 - Gwx + Lx^2 &= 0 \end{aligned}\right\}$$

is

$$\cos^{-1}\frac{D}{2\sqrt{AB}} + \cos^{-1}\frac{E}{2\sqrt{BC}} + \cos^{-1}\frac{K}{2\sqrt{CA}} + \cos^{-1}\frac{G}{2\sqrt{LA}} = 0,$$

and therefore from section 16 is

$$\begin{vmatrix} \frac{D}{2\sqrt{AB}} & \frac{E}{2\sqrt{BC}} & \frac{K}{2\sqrt{CL}} & \frac{G}{2\sqrt{LA}} + \frac{DEK}{4BC\sqrt{LA}} \\ \frac{E}{2\sqrt{BC}} & \frac{D}{2\sqrt{AB}} & \frac{G}{2\sqrt{LA}} & \frac{K}{2\sqrt{CL}} + \frac{EDG}{4AB\sqrt{CL}} \\ \frac{K}{2\sqrt{CL}} & \frac{G}{2\sqrt{LA}} & \frac{D}{2\sqrt{AB}} & \frac{E}{2\sqrt{BC}} + \frac{KGD}{4LA\sqrt{BC}} \\ \frac{G}{2\sqrt{LA}} & \frac{K}{2\sqrt{CL}} & \frac{E}{2\sqrt{BC}} & \frac{D}{2\sqrt{AB}} + \frac{GKE}{4CL\sqrt{AB}} \end{vmatrix} = 0 .$$

Multiplying the rows in order by $4BC\sqrt{LA}$, $4AB\sqrt{CL}$, $4LA\sqrt{BC}$, $4CL\sqrt{AB}$ we change this determinant into

$$\begin{vmatrix} 2DC\sqrt{BL} & 2E\sqrt{ABCL} & 2KB\sqrt{AC} & 2GBC+DEK \\ 2EA\sqrt{BL} & 2D\sqrt{ABCL} & 2GB\sqrt{AC} & 2KAB+EDG \\ 2KA\sqrt{BL} & 2G\sqrt{ABCL} & 2DL\sqrt{AC} & 2ELA+KGD \\ 2GC\sqrt{BL} & 2K\sqrt{ABCL} & 2EL\sqrt{AC} & 2DCL+GKE \end{vmatrix},$$

and now dividing the columns in order by $2\sqrt{BL}$, $2\sqrt{ABCL}$, $2\sqrt{AC}$, 1 we have finally

$$\begin{vmatrix} DC & E & KB & DEK+2GBC \\ EA & D & GB & EDG+2KAB \\ KA & G & DL & KGD+2ELA \\ GC & K & EL & GKE+2DCL \end{vmatrix} = 0 ,$$

which agrees with what has been obtained otherwise.*

* *Proc. Roy. Soc. Edin.*, xxi. p. 333.

XI.—*Chapters on the Mineralogy of Scotland. Chapter VIII.*—Silicates.* By M. Forster Heddle, M.D., Past President of the Mineralogical Society of Great Britain, Emeritus Professor of Chemistry in the University of St Andrews.

(Read December 6th, 1897.)

The earlier mineralogists laboured under two great disadvantages. They could not readily, on account of the small number of students of chemistry, call in the aid of that science: and at the time when mineralogy was becoming a distinct branch of science chemistry was in itself crude as well as cumbrous. They were thus forced to rely chiefly upon external properties; and, where crystalline form was absent, they were confined to what may be called physical properties alone.

Their knowledge of the composition of bodies being thus limited and uncertain, the old nomenclature was to a considerable extent founded upon external features alone.

It is the habit of many of the silicates to run out into lengthened crystals, the greatest amount of their concreting material being deposited in the direction of the main axis of the crystal, and when a multiplicity of crystals are concreted, these are thrown out from a common centre of crystallising growth, to radiate through the matrix, very much after the manner of such crystals as have grown in what we term empty or free space, where no matrix is present to interfere with a tendency to divergence. This fact, the evident displacement of that which is not now displaceable, gives us, in the first place, some information as to the condition of the matrix of divergent crystal groups at the time of their formation; and leads us, in the second, to consider whether that matrix was in a very different condition, or held in degree any very different relationship (as a *body foreign* to the substance crystallising in it) from the liquid or the vapour present in those cavities in which we usually find divergent crystalline groups.

The Swedish mineralogist Wallerius, who wrote in 1747, was one of the earliest authors who instituted group-arrangements. After considering the gems, and rock-

* Chapter I. The Rhombohedral Carbonates. Part I., . . . *Trans. R.S.E.*, vol. xxvii. p. 493.
" II. The Felspars. Part I., ... " " xxviii. p. 197.
" III. The Garnets, ... " " xxviii. p. 299.
" IV. Augite, Hornblende, and Serpentinous Change, . . " " xxviii. p. 453.
" V. The Micas; with description of Haughtonite, a new Mineral Species, ... " " xxix. p. 1.
VI. "Chloritic Minerals," . . . . . " " xxix. p. 55.
VII. Ores of Manganese, Iron, Chromium, and Titanium, . " " xxx. p. 427

species, such as felspar, mica, talc, and asbestos, he instituted a family which he termed *Hornbärg* (the *Roche de Corne* of the French); this embraced, as a sub-family, Skiörl (Schörl of the Germans).

Cronstedt (1758) adopted the same family, and threw into it, "as a convenient pocket," many of the recently discovered species.

It is not altogether easy to make out what species were entitled to get into this pocket, or which side of its medial partition it was intended that they should lodge themselves in, beyond this that *corneus* was to hold "cheap or worthless stones," "mostly of colours from black to dull green."

Cronstedt introduced a little more method as regards the "*Skorl*" side of the pocket, making it nearly synonymous with the *corneus crystallisatus* of Wallerius, and destined to receive *prismatic minerals* of black, brown, green, and reddish colours, but still having some resemblance to horn in lustre.

He, however, again introduces confusion—a confusion which was continued by Wallerius in an edition of 1778—through adopting the term "Basaltes" instead of Skorl; and Hill, in his work on *Fossils*, 1771, fortifies the error, when, in speaking of the *Shirls*, he says, "as to size we see them from that of barleycorn up to the Giant's Causeway," the columns of which he calls "Basaltes Hibernicus" or "Irish Shirl."

Romes de Lisle, however, in 1783, bringing crystallography to bear, at once got rid of such excrescences as basaltic pillars, helleflintas, and rocks; but on account of chemistry being still a lagging science, he was forced to throw in many new and indeed old species, and to increase the number of adjective distinctions; and though we do find these to be all Silicates, yet he departs from the prismatic elongation by introducing such species as axinite, staurolite, and harmotome.

When chemistry came to lend a hand to the structural erection of the science, the disintegration of the great "schorl group" commenced. Bergmann, by his researches, published in 1780, went far to disband it; the five which lingered last were kyanite, "blue schorl"; staurolite, "cruciform schorl"; andalusite, "a red schorl"; rutile, red schorl; tourmaline, black schorl; and these were extruded from the family in the above order. There is this much indication of these forming a natural group that, with the exception of rutile, they are all silicates of alumina.

It is somewhat singular that the substance—namely, tourmaline—which last retained the name schorl seems to have been that to which the term was first applied. In Matthesius' *Sarepta*, 1562, we find that the name "schurl" was used for the "sterile black little stones" accompanying tin ore and gold, and which were thus probably *tourmaline*; and as they were metallurgically worthless, it has been suggested that the word originally was derived from the old German word *Schor*, meaning *refuse*.

"Schorl" is a name still applied to an inferior fibrous, opaque, black tourmaline; it is the sole representative of a great family; but we still use the adjective *schorlous*—as "schorlous beryl"—to imply crystals, thinner and more elongated than usual, which are *imbedded in a matrix with more or less of a radiating arrangement.*

It is more especially the minerals retained longest in this old family of the schorls which fall to be considered in the present chapter, and in chemical simplicity the first of these is Andalusite, $\dddot{\cancel{\text{Al}}}\ \dddot{\text{Si}}$, right prismatic.

1. *Red Andalusite, from Auchendoir, Aberdeen.*

"Red Schorl" from Aberdeenshire has been noticed in several old works, but Mr JAMES SOWERBY, the author of *British Mineralogy* (1804), has the credit of first *describing* the mineral. He does this with precision, but, though he shows what it is not, he draws a false conclusion as to what it is.

His description is as follows:—

"Argilla durissima, *Scotch Corundum*, spec. char. Nearly pure argil; hardest of all minerals, next to the diamond.

"This curious substance was sent me from a dealer in Aberdeen, under the name of Red Schorle from Achen-door. I figure it here because it is a substance which appears to be new to British writers. Upon inquiry I found it was very little known, nor was it to be found in any mineralogical collection in London, nor scarcely in Scotland. Even Mr JAMESON had not previously obtained it. From him I hope for a good account of it."

Then follows his description, which concludes: "Among a tolerable quantity I found very few with crystallised terminations; the faces, however, are very distinct. We find this fossil has been taken for a rubellite, and KIRWAN'S description in a great measure accords with that idea. But in many respects it has been confounded with the titanite of KIRWAN. May the radiating variety be the substance of which MACQUART says the garnets are formed? He describes it as consisting of straight fibres diverging from a common centre. KIRWAN mentions red schorl, and says rubellites are so called. Another substance resembling this was found by MORVEAU in Poitou, which he presumed to be adamantine spar."

After showing how it differs from certain of the above, and giving its properties, SOWERBY writes: "This seems undoubtedly the 'Spath adamantin d'un rouge violet' of BOURNON, which he now considers a variety of corundum."

SOWERBY finally points out JAMESON'S mistake as to corundum occurring at Tiree,* and concludes, "therefore, ours is the only thing known at present as corundum in Scotland."

Though Auchendoir has been given as the locality for this red andalusite, I rather think that both the localities in which I myself found it are in the parish of Kildrummy, and that only the grey variety is found in Auchendoir. These localities lie a few miles to the south-west of the village of Lumsden. The first—the south side of the Peat Hill—affords but few specimens, and these are poor. The second is the southern slopes of the hill of Clashnaree, in Clova.

* The Tiree mineral is greyish-white malacolite.—M. F. H

The specimens all lie loose, being the most enduring portions of veins which have themselves endured after the disintegration of a very micaceous gneiss. To all appearance, indeed, the specimens seem to represent the "branches" or knots upon very thin veins of quartz, which veins can, after considerable search, be seen formed in the rock, and such as I have found were barren of minerals. The loose lying fragments of veins consist of a melange of quartz, andalusite, labradorite, fibrolite, and an ill-defined black mica.* These minerals interlace in a confused manner, there being no approach to a uniformity in growth from the two sides of the vein. The andalusite crystals, indeed, sometimes pass from side to side, lacing the other ingredients together. The mineral is always rudely crystalline, but regular crystals are very rare. The most perfect I have delineated.

Though I have figured them as "complete" in the terminal planes, yet all the crystals I have seen had these planes hemihedrally disposed. The colour is a uniform dull purplish red; but there is this most important fact to be noted, that all the crystals which can be sectioned and examined, though uniform in structure and transparent in thin slices, have a central core which is deep purple, with purple spots at the four corners of the transverse section, after the manner of chiastolite. Well-crystallised andalusite thus seems to have a complex internal structure which is independent of any portion of the matrix being caught up during its concretion into a geometric solid. This fact, not, so far as I know, before noticed, comes to have an important bearing in all speculations as to the question of the mode of formation of chiastolite crystals in clay-slate rocks.

The crystals are sometimes 3 or 4 inches in length, and occasionally an inch in thickness. Rarely, as noticed by SOWERBY, they form a tube-like sheath to a central core of the felspar; there is not here the slightest appearance of any passage into felspar, as assumed by BOURNON; but there is an almost insensible passage into, or intermixture between it and colourless and brilliant lustred fibrolite.

The specific gravity of this red andalusite is 3·121.

The analysis on 1·302 grammes yielded—

Silica,	·442	
from Alumina,	·036	
	·478	= 36·712
Alumina,		59·678
Ferric Oxide,		2·302
Manganous Oxide,		·230
Lime,		·860
Magnesia,		trace
Water,		·465
		100·247

Insoluble silica, 4·184 per cent.

* See Chapter V. The Micas (*Trans. R.S.E.*, xxix. (1879) 33).

2. *Andalusite from Marnoch, Banffshire.*

The precise locality is the banks of the stream near the Mill of Achintoul, Kinnordy Castle.

The nature of the ground at Clashnarec—for it is covered with sward and peat—and the consequent impossibility of tracing the specimens into connection with the rock, as well as the confused crystalline arrangement of the constituents of the veins, prevents our arriving at any information which can have a geologic bearing on its formation, simple though it be in composition. This should not be the case, however, as regards its occurrence at Marnoch and elsewhere in Banffshire; though the light thrown therefrom is still obscure. And yet it is not so much that the amount of evidence supplied by the mode of occurrence and internal structure of such substances as the andalusite of Marnoch, the staurolite of Aldernie, the chiastolite of Portsoy, the apophyllite of Kilsyth, and the stilbite of the Long Craig is in itself small, as that we have collected so few observations on the paragenetic formations, and know so little of the physical laws which govern the formation of such crystals as are built up, not according to ordinary polar molecular concretion, but apparently by the sequential interlocking of tesselated structures, each one of which structures seems to have been constructed in defiance of all the recognised laws of crystalline accretion.

As regards internal structural arrangement—the mode of fitting of the molecular or crystallo-molecular bricks of the fabric—the imbedded andalusite crystals of Marnoch yield almost no insight. Because in the formation of the crystals—however that was effected—so much of the matrix has been caught up by the concreting andalusite substance as must be regarded as capable of interfering with the free formation of any definite structure, seeing that it has chemically interfered with the purity of the material attempting to crystallise apart. Possibly its potency to interfere may be all the greater that the intruding substance is present not in the condition of a *magma mica*, but as a perfected mineral formation—biotite mica.

The crystals of andalusite at Marnoch lie all imbedded in a fine-grained schist, which has, when fresh, a pale yellow-brown colour, due to a crypto-crystalline magma of silicate of alumina. This magma is sprinkled throughout with minute crystals of rich brown biotite, granules of quartz, specks of magnetite, and twin crystallisations of staurolite, of less than pin-head bulk. The crystals of biotite lie in all directions, pervading the whole mass; those of staurolite have some disposition to be arranged in special layers; and this is very much more marked as regards the crystals of andalusite, though there are other localities in which it is hardly observable.

The crystals of andalusite are from half to nearly one inch in length, by about one-third of that thickness, and it is to be remarked that though for the most part here disposed in layers, they are very far from invariably disposed upon their sides, as regards the rock bedding, though that position dominates. They are ash-grey in colour, and in section and even to the eye a central lozenge-shaped tessela of darker and clearer shade of colour is seen; while the whole substance of the crystals is also seen to spangle with crystals of biotite. These are equal in size to those generally occurring in the rock, are disposed like these in all directions, and are not very markedly fewer in number.

The analysis of these was made on crystals freed absolutely from the inclosing rock, and with even some portion of their outer surfaces removed to ensure as great purity as possible. They yielded—

On 1·3 grammes—

Silica,	52·538
Alumina,	39·314
Ferric Oxide,	1·094
Ferrous Oxide,	3·267
Manganous Oxide,	·461
Lime,	·861
Magnesia,	·846
Alkalies,	trace
Water,	1·11
	99·491

Loss, ·238 per cent. of water in the bath.

This result shows a very considerable intermixture with all of the ingredients of the rock, notwithstanding which the crystals are hardly affected by the knife, and have a vitreous lustre.

Three theories have been advanced to account for the presence of the crystalline constituents of clay-slates, for they occasionally bulk so largely as to entitle them to the name. According to the first of these theories, the crystals in question are regarded as the product of chemical action in the ocean in which the original material was deposited. The second theory attributes and confines the formation of the crystalline minerals to processes of metamorphism which have taken place subsequent to the solidification of the rocks. The third theory refers them to an aggregative action going on in the still plastic clay-slate mud prior to its solidification.

The first of these theories has been maintained by Credner; but against it numerous arguments have been adduced, and especially the difficulty of supposing an ocean capable of depositing from its waters at successive periods minerals of such different chemical composition as actinolite, andalusite, chlorite, etc.

The second theory has received the support of DELESSE, but in opposition to it the existence in the rocks in question of broken crystals which have been re-cemented by the surrounding clay-slate substance has been pointed to.

Striking facts, drawn from the microscopical structure of the rocks, have been adduced by ZIRKEL in favour of the third theory.

Later metamorphic action must not, however, be excluded in seeking to account for the origin of the crystalline constituents of clay-slates.

A review even of the theories themselves suffices to show that four distinct stages at least may be considered in the series of changes by which the rocks in question may have acquired their present character :—

1*st*, the deposition of the mud ;

2*nd*, the formation of minerals during the plastic state ;

3*rd*, the separation or segregation of other materials after solidification ; and

4*th*, the action of metamorphic processes.

If such processes have operated locally, it will have to be considered whether they most favour the second or the third of these theories, for they may be *local* in their operation either geographically or geologically. They may have operated in close proximity to igneous outbursts, or to limestone formations where there has been much crushing of the beds, or even when there has been disturbance alone. And, geologically, the change may be apparent throughout the whole sweep of a formation, but only up to a certain thickness of its deeper-seated beds.

TOURMALINE.

This substance, common in granitic veins as it is, does not often occur in Scotland either in well-developed forms or of marked purity. The finest crystal I know of, the terminal portion of which I examined, was found in the coarse granite vein of Rubislaw quarry. It occurred along with microcline, muscovite, beryl, and garnet. It was 8½ inches in length by 1¼ in width. It was curved like the figure 6, but was perfectly terminated and formed throughout. Fine crystals are rarely found in granite veins in andalusite schist in North Glen Clova in Aberdeenshire.

Material sufficiently pure for analysis was prepared from several localities, but our want of any satisfactory method of determinating boracic acid induced the writer to postpone the analyses, except in the case of crystals which were found in the granitic belt of rock which cuts gneiss near Struay Inn, Ross-shire.

It here occurs in jet-black crystals of some inches in length along with muscovite, orthoclase granular pink, and microcline of a dove blue, garnet and beryl. Its specific gravity is * . In powder it is brown.

* The blank was in the MS. Professor GEIKIE informs me that the specific gravity lies between 3·1 and 3·24 ; Scottish examples being nearer to the latter than to the former value.—P. G. T.

On 1·3 grammes—

		P.C.
Silica,	·457	
from alumina,	·005	
	·462 =	35·538
Alumina,		35·55
Ferric Oxide,		·18
Ferrous Oxide,		7·12
Manganous Oxide,		·307
Lime,		1·108
Magnesia,		3·538
Potash,		1·072
Soda,		·429
Boracic Acid (loss),		10·768
Fluorine,		1·705
Phosphoric Acid,		trace
Water,		2·955
		100·000

Fibrolite, $\cancel{\text{Al}}\,\ddot{\text{S}}\text{i}$. Anorthic.

This species was first recognised as British by the writer, but there is reason to believe that it was noticed by SOWERBY, although he was ignorant of its true nature.

In speaking of the andalusite of Auchendoir, while stating that it does not merge into felspar, he remarks: "The nearest approach to mixing insensibly is by fibres, which in ours are, however, sufficiently distinct." He also remarks: "The gangue is chiefly composed of a coarse granite intermixed with indurated asbestos."

In the first, if not in the second of these observations, he must refer to fibrolite, and had he laid due weight upon the fact that the fibres were "sufficiently distinct," he would have seen that they must have been a material different from the andalusite which he was describing.

The fibrolite of Clashnaree occurs in three different modes of arrangement. First, as a corded or stalactitic-like coating to the other minerals, somewhat after the manner in which galmei coats galena. Here it forms a kind of sheath which envelopes labradorite, quartz, and andalusite alike. Second, it radiates in bundles of fibres through the labradorite, and these fibres often unite into a mass which resembles okenite. This variety is very tough. Third, it frequently is disposed with its fibres in parallel arrangement to the crystals of the red andalusite; and long slender crystals of the red andalusite are often imbedded amongst the fibres of the fibrolite.

As the fibrolite is white or colourless, and of adamantine lustre, it is easily distinguished, and there is nothing of the nature of a transition; it is a case of the main

axis of dimorphous substances lying parallel to one another, as known to occur with grenatite and kyanite, and with other di-morphs.

In this third form it is somewhat more brittle than in the others, but it is still reduced to powder with extreme difficulty. I with difficulty separated a sufficiency of the fibrolite in its third form for analysis; but when separated it was exquisitely pure and brilliant. It had a hardness fully 7· in the scale.

22·1 grains yielded—

Silica,	38·410
Alumina,	61·426
Ferric Oxide,	·215
Manganous Oxide,	·114
Water,	·23
	100·395

3. *Fibrolite from Pressendye Hill, Tarland, Aberdeenshire.*

The specimens examined I found in small quantity coating gneiss, in thin veins on the north-west side of the hill, at about 300 yards from its summit.

Its colour was dull white; it was not very lustrous; it was in fibrous and slightly matted tufts, which were very tough. No piece was got large enough for the determination of the specific gravity.

It yielded—

Silica,	39·680
Alumina,	58·822
Ferrous Oxide,	·038
Manganous Oxide,	1·100
Potash,	·860
Soda,	trace
Water,	·320
	100·820

Dr Thomas Aitken of Inverness showed me fragments of granite boulders which he had collected at Auchendown, near Cawdor. These contain a substance of an appearance very similar to the last. There is, however, some suspicion in my mind that this may be merely somewhat plicated plates of a hydrous mica, which show the edges of the plates only. The specimens, having been exposed, are not altogether fresh.

There is one fact which so far increases the probability of this being fibrolite, namely, that a black mica, which has much the appearance of that associated with the mineral at Clashnaree, is present in the Auchendown boulders.

Kyanite, ~~Al~~ Si. Anorthic.

There is good reason to believe that this species was first found in Scotland.

SAUSSURE, *fils*, describes it under the name *Sappare*, in *Journ. de Phys.*, xxxiv. 213, 1789. His name *sappare* arose from a mistake in reading a label of the mineral, on which it was called *sapphire*; a copy of this label is given in the *Journ. de Phys.* The specimen thus labelled was from Botriphnie in Scotland, and was sent by the Duke of GORDON to SAUSSURE the father.

In the *Descr. Cat. de l'École des Mines*, p. 154, published by SAGE in 1784, it is called *Talc bleu*; but as the present writer found no "Talc bleu" in the collection of l'École des Mines, and as he among the specimens of kyanite found a Botriphnie specimen of the mineral, it is probable *that* had been the specimen termed the "Talc bleu" by SAGE, and the specimen presented by SAUSSURE, having come from the same original source.

The name *sappare* was used for the mineral by some writers up to 1823, when we find it employed by JAMES SMITHSON who, in virtue of its infusibility, used it as a support in blowpipe experiments.*

Kyanite is no longer got at Botriphnie, and the precise spot where it occurred I have not been able to find. Specimens from this locality are in the collection at Jermyn Street Museum, and in those of Edinburgh, Banff, and, I think, Montrose. They were larger and finer than any now obtained in Scotland. The only associated mineral is margarodite.

The second locality at which this mineral was found in Scotland was in the vicinity of the Burn of Boharm, about a mile above the house of Auchlaukart—that at least is the spot where the writer has found it in *North* Boharm.

Dr MACCULLOCH, in writing of it at this spot, gives the following accurate description of it, one which should be pondered in considering the metamorphism of the rock matrix.

"Boharm. This sappare-disthene is said to have been originally discovered in this place. The crystals occur in a quartz vein which traverses a talcose clay-slate. They pass through both without any change of their direction or appearance; seeming to mark a common condition in the schist and the quartz at the period of their formation. Although these crystals in general penetrate and impress the quartz, they are sometimes bent and waved, as if they had accommodated themselves to its irregularities. This is not the case, however, with those imbedded in the talcose slate, which radiate in brushes of rectilinear crystals through its mass. This rock consists of a talcy clay-slate, so penetrated with hornblende as to render its character for an instant doubtful. On an accurate examination it will be seen that the body of the rock is a clay-slate, and that it is interspersed throughout with lamellar and thin crystals of hornblende. These lamellæ are generally disposed at right angles to the lamella of the schist, and are sometimes short and straight, and variously placed, interfering with each other often in every direction. More commonly they diverge from a sort of central axis in curved planes, so

* SMITHSON remarks: "Chemical analysis carries destruction along with it, and bestows knowledge of a substance only at the cost of its existence. One remedy which can be offered for this defect is to reduce the scale of operating, and thus as far as possible reduce the amount of the sacrifice."

that their section, according with that of the lamella of the schist, exhibits an appearance of curved pencilliform groups of acicular crystals, frequently an inch in length, assuming an appearance of great singularity. In this direction the schist is visible, and appears to form the largest part of the stone, while in the cross fracture, the lamellæ of hornblende alone being seen, the whole rock seems to consist of this mineral. Occasionally the hornblende displays crystals disposed in so many different ways that the schist is discernible even in the cross fracture."

To this description I have only to add that the specimens I have obtained from near Auchlankart were all of the *rhætizile* or grey variety, much impregnated with the substance of the schist, in which indeed I alone here found them, but that I found at the same spot—which is at the upper fork of the burn—crystallised staurolite in simple crystals, the mode of the occurrence of which—as regards the quartz and the rock matrix which alike hold them—was precisely as described by MACCULLOCH for disthene. These crystals of staurolite were amber coloured and transparent, but had a central structure, which will be noticed below.

Specimens nearly as fine as those from Botriphnie were formerly found by Colonel IMRIE loose lying in the neighbourhood of Millden and the Burn of Turret, North Glen Esk, Forfarshire. One of these has been figured by SOWERBY, vol. iii. p. 49. Here also margarodite is the sole associate.*

"Near Banchory, in Aberdeenshire," "near Mortlach, Banffshire," and "in quartz near the summit of Ben y Gloe," in blue radiating crystals, in quartz nodules, in clay-slate, in limestone at Ardonald, by Cunningham, are old localities at which this mineral is no longer found.

It has long been known, and is still found at Vanleep, Hillswick, Shetland. At this gash, a chasm in the cliffs of the western shore of Hillswick, kyanite occurs of three markedly dissimilar appearances.

The ordinary blue crystals generally isolated and imbedded in massive quartz are here very rare. Large plumose groupings of a reddish-grey colour, also occurring isolated in massive quartz, are less rare; but the common appearance is that of veins or large isolated nodules of smaller intermatted crystals of an anchovy-red passing into

* I analysed a specimen from Colonel IMRIE's collection, and obtained on 1·3 grammes :—

Silica,	36·384
Alumina,	58·296
Ferric Oxide,	1·609
Ferrous Oxide,	1·123
Lime,	·861
Potash,	·252
Soda,	·423
Water,	1·445
	100·393

The loss in bath was	·282 per cent.
The insoluble silica,	1·691 ,,
The specific gravity,	3·538 ,,

white, and apparently dark green, from an intimate intermixture of chlorite plates. Occasionally a plate or two of talc occurs, and very rarely large and fine crystals of chloritoid. These veins cut the huge beds of quartz which intercept the micaceous strata of the promontory. The locality faces the picturesque sea-stacks of red porphyry termed the Drongs. The crystals analysed were picked white, somewhat tinted with pink.

On 1·2 grammes—

Silica,	·474	
from Alumina,	·022	
	·496	= 38·153
Alumina,		56·979
Ferric Oxide,		1·867
Manganous Oxide,		·153
Lime,		·301
Water,		2·646
		100·099

Loses in the water-bath, ·701 per cent.
Insoluble silica, 3·024 per cent.

From near Millden in Tarffside, Forfarshire. This occurs in large flat crystals of a fine blue colour.

I have found it at the following new localities in Shetland. Cliffhill, near Woodwick, and north-west of Norwick Bay in Unst. *Magnetite* and *garnets* are its associates at the first of these localities; it is in quartzose belts at the last; the rock in both cases being gneiss, and the colour of the mineral pale blue. At the south end of the Wark of Skewsburgh, in the Mainland, associated with *ilmenite* in quartz veins in gneiss. It is here greyish-white to blue. To the east of the same hill near its north end.

Kyanite has more recently been found at the following localities :—

In minute crystals of perfect transparency and deep blue colour along with green hornblende and red garnet, forming the rock eklogite. This was found by Mr Dudgeon, to the north-east of Obb, in Harris.

Finely crystallised in the form of the figure and of a fine blue colour, at a height of about 1100 feet, on the north-west slopes of Garlat Hill, Cowie Hill, Tarffside, by Mr Robert Murray. The matrix here was gneiss and the associate finely crystallised chlorite.

In interlacing grey crystals in gneiss far up in bed of the burn which comes from the east into Glen Derry, Loch Callater, Aberdeenshire, by the Rev. Mr Peyton.

In blue crystals in gneiss in Allt Beg, Glen Rinnies, by Mr James Wilson.

In mica schist at a bridge over the Little Drumiach, in the parish of Enzie, Banffshire,* by Mr Wallace of Inverness.

* *Min. Mag.*, vol. vi. No. 28. But no description given.

By the writer it has been found—

In small blue crystals in Hebridean gneiss on the hill to the west of Ben Chaipaval in Harris.

In quartzite near summit of Carn Lia, Ben y Gloe, Perthshire.

In greyish-blue crystals, along with garnet, sphene, ilmenite, and chlorite, in mica schist, one mile north of Loch Bulg in Aberdeenshire.

In large blue-grey crystals, along with ilmenite and chlorite, in gneiss on the slopes on the east side of the corry of Meall Buidh, on the south side of the Moor of Rannoch.

In gneiss in the railway cutting west of Mulben, Banffshire.

In bright blue crystals in gneiss near limestone about one mile west of the limestone quarries at Dulnan, Inverness-shire.

Loose lying in grey and blue interlacing crystals on the south slopes of Cruach Ardran, in Perthshire.

In tufts of grey crystals impregnated with the substance of the rock in clay-slate, at the lime quarries of the Burn of Aldernie, Banff.

In large single imbedded blue crystals and fasciculitic tufts, a peculiar yellow margarodite slate, south-east of the lime quarries in Glen Urquhart.*

In quartzose veins in a clay slate which contains rosette groupings of actinolitic crystals in the rocks, about three-quarters of a mile north-west of Sandend in Banffshire. The crystals of the mineral here, an inch or two in length, pass through portions of both matrix and vein, after the manner of rivets, just as described by MACCULLOCH as occurring at Boharm. They appear to have issued from the matrix into the vein, as if formed nearly contemporaneously with the filling of the latter with the quartz; but as the terminations are not distinct, this conclusion is drawn from the greater breadth of the crystals, where they lie in the quartz, than in the schist.

These crystals, like those at many other of the above localities, are in parts colourless or pale yellow.

In lenticular quartz nodules, often morion, and sometimes prase, with pyrite, in chiastolite slate, west of the clay-slate quarry near Portsoy, sometimes colourless.

* This yielded on 1·3 grammes :—

Silica,	37·53
Alumina,	58·105
Ferric Oxide,	2·069
Lime,	·129
Magnesia,	·076
Potash,	·252
Soda,	·741
Water,	1·198
	100·12

Its specific gravity was 3·016

EPIDOTE.

1. From Balta Island, Shetland. Occurs in a geo which cuts the island near the centre of its eastern cliff-lined shore. The epidote occurs in crystals of half an inch in size, imbedded without any associate in a quartz vein which cuts gabbro. The epidote is pale pea-green, the quartz somewhat granular.

On 1·396 grammes—

Silica,	·531	
from Alumina,	·01	
	·541	= 38·753
Alumina,		26·986
Ferric Oxide,		7·898
Ferrous Oxide,		1·806
Manganous Oxide,		·501
Lime,		20·378
Magnesia,		·786
Potash,		·25
Soda,		·21
Water,		2·376
		99·944

Insoluble silica, 7·578 per cent.

2. The above occurrence of epidote certainly in no way bears out the theory of its resulting as a product of the alteration of hornblende. This, however, may have been the case at the next locality which I quote. This is Nudista in Hillswick.

It here occurs in dull, soft-looking crystals of an olive-green colour, of about an inch in length. These crystals radiate through large foliated crystals of dark green hornblende, cutting the foliations transversely. The specific gravity of this epidote is 3·396.

On 1·304 grammes—

Silica,	·484	
from Alumina,	·009	
	·493	= 37·866
Alumina,		24·722
Ferric Oxide,		9·961
Ferrous Oxide,		·361
Manganous Oxide,		·536
Lime,		23·104
Magnesia,		·766
Water,		2·822
		100·138

3. From North Quin Geo, Hillswick. Epidote occurs here filling a small vein. It forms stellate groups of rich green crystals over an inch in length.

The analysis on 1·503 grammes yielded—

Silica,	36·127
Alumina,	20·574
Ferric Oxide,	14·921
Manganous Oxide,	·306
Lime,	23·025
Magnesia,	·306
Water,	4·568
	99·827

Insoluble silica, 7·734 per cent.
The ferric oxide is to the alumina as 1 to 2.

4. From Delnabo, Glen Gairn, Aberdeenshire. Out of the old limestone quarry. It occurs imbedded in green prehnite, in radiated crystals of a pale green colour.

On 1·501 grammes—

Silica,	38·374
Alumina,	26·087
Ferric Oxide,	10·388
Manganous Oxide,	·738
Lime,	21·647
Magnesia,	·239
Water,	2·441
	99·914

Insoluble silica, 7·812 per cent.
The ferric oxide is to the alumina as 1 to 4.

WITHAMITE.

This red variety has been found only at one spot. This is a projecting spur of amygdaloidal felspathic porphyry, which touches the road through Glencoe upon its north side, about three miles above the turn of the glen. The epidote occurs in the little druses, very rarely in bright green crystals. It is then associated with byssolite and chlorite. Much more frequently it occurs in the red modification. It forms very minute acicular crystals of a brilliant blood-red colour. These crystals radiate from the sides of the druses, a narrow layer of a milky saussurite-like substance sometimes intervening. The crystals are red and yellow respectively in two directions, at right angles to one another (MacKnight and Brewster). Minute specks of quartz sometimes occur.

On account of the extremely minute quantity in which this substance is found, the purifying of the sample analysed was executed with extreme care.

On 1·3 grammes—

Silica,	·543	
from Alumina,	·019	
	·562 =	43·23
Alumina,		23·09
Ferric Oxide,		6·675
Ferrous Oxide,		1·131
Manganous Oxide,		·138
Lime,		20·003
Magnesia,		·884
Potash,		·962
Soda,		·935
Lithia,		·253
Water,		2·4
		99·701

Insoluble silica, 1·957 per cent.

This result is by no means a satisfactory agreement with the composition of epidote.

ZOISITE.

1. This mineral was first found by me in Britain in Glen Urquhart, Inverness-shire. It occurred in the most southerly of the limestone quarries, about a mile north-east of Milltown. It was found only in one large nodule of calcite of about a hundredweight, crystals of a grey to a pale bluish-white colour, about one inch in length, interlaced in the calcite. There was a very little quartz; a few specks of chalcopyrite and brushes of light green actinolite in association with the mineral.

The crystals, and indeed the mineral, has the general appearance of tremolite, but the cleavage leaves no room for doubt. The form is well seen in these crystals, but there are no terminations. The cleavage face seems a twin face, as there are repeated re-entering angles which produce a coarse striation. The crystals are brittle. The cleavage face is somewhat pearly; fractures are vitreous. The specific gravity taken on three pieces gave 3·004, 3·111, and 3·014.

On 1·303 grammes—

Silica,	·484	
in filter,	·019	
from Alumina,	·013	
	·516 =	39·60
Alumina,		31·083
Carry forward,		70·683

Brought forward,	70·683
Ferrous Oxide,	2·071
Manganous Oxide,	·078
Lime,	23·336
Magnesia,	trace
Potash,	·566
Soda,	1·056
Water,	2·412
	100·202

Insoluble silica, 1·55 per cent.

2. From an adjacent quarry; the mineral being very similar, but the matrix was quartz. The specific gravity is 3·014.

On 25 grains—

Silica,	41·56
Alumina,	29·901
Ferrous Oxide,	3·205
Lime,	22·142
Magnesia,	·332
Potash,	·345
Soda,	·684
Water,	2·19
	100·359

Insoluble silica, 2·31 per cent.

3. From the same quarries, but in large crystals of about 2 inches by $1\frac{1}{4}$. These large crystals were lying loose in the quarry, the calcite having apparently been dissolved away by rain. They were slightly browned on the outside, but lustrous when broken. Three determinations of the gravity gave 3·312, 3·322, 3·318.

On 1·301 grammes—

Silica,	·48	
from Alumina,	·034	
	·514	= 39·508
Alumina,		30·827
Ferrous Oxide,		2·52
Manganous Oxide,		·077
Lime,		22·813
Potash,		·681
Soda,		·9
Water,		2·505
		99·831

Insoluble silica, 2·334 per cent.

4. From Laggan, Dulnan Bridge, Inverness-shire.

This was given me by Sir ARCHIBALD GEIKIE. It was got in quartz veins in the limestone quarry. It occurs in pale brown crystals entirely imbedded in the quartz. The crystals are lustrous and well defined; the associates are chlorite and sahlite. In the near neighbourhood there is much kyanite. The specific gravity is 3·438.

On 1·2 grammes—

Silica,	·463	
from Alumina,	·002	
	·465 = 38·75	
Alumina,		28·144
Ferric Oxide,		6·547
Manganous Oxide,		·916
Lime,		22·026
Magnesia,		·416
Water,		3·333
		100·132

Loses in the bath, ·155 per cent.

5. From Loch Garve, Ross-shire. This locality was found by the late W. H. BELL. It is that in which the mineral occurs in much largest quantity in Scotland, and also in much the largest crystals. It occurs in a quartzose vein, which, starting from near the centre of the south shore of the lake, strikes right up the hill for two or three hundred feet. Sometimes it is almost massive, occasionally in crystals of a stouter habit than those of Urquhart. The colour is ash-grey to white, passing to pale yellowish-brown. It is translucent. The specific gravity is 3·268.

On 1·34 grammes—

Silica,	40·066
Alumina,	30·834
Ferric Oxide,	1·58
Ferrous Oxide,	·48
Manganous Oxide,	·22
Lime,	23·66
Magnesia,	·476
Potash,	·504
Soda,	·428
Water,	2·100
	100·348

Insoluble silica, 2·2 per cent.

IDOCRASE.

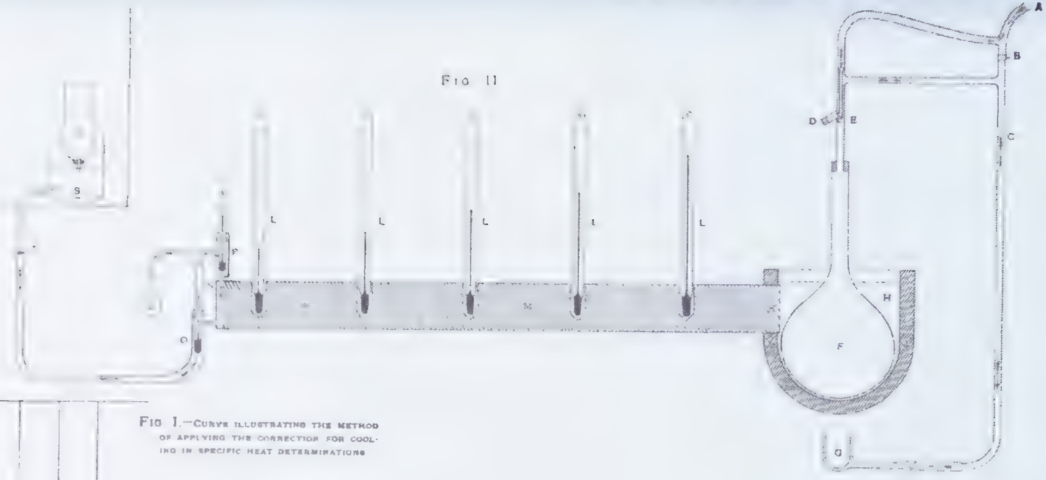
From Delnabo, Glen Gairn, Aberdeenshire. Idocrase occurs abundantly, passing when massive almost insensibly into garnet in the old limestone quarry. The portion taken for the analysis was a portion of a magnificent dark brown crystal of about 7½ inches in length by nearly 1 inch in width, which was fractured by the blow which disclosed it. Its specific gravity was 3·43.

On 1·302 grammes—

Silica,	·45	
from Alumina,	·022	
	·472 =	36·251
Alumina,		18·626
Peroxide of Iron,		·932
Ferrous Oxide,		5·036
Manganous Oxide,		·844
Lime,		33·935
Magnesia,		1·574
Potash,		·568
Soda,		·329
Water,		1·78
		99·875

Insoluble silica, 1·483 per cent.

Fig. II



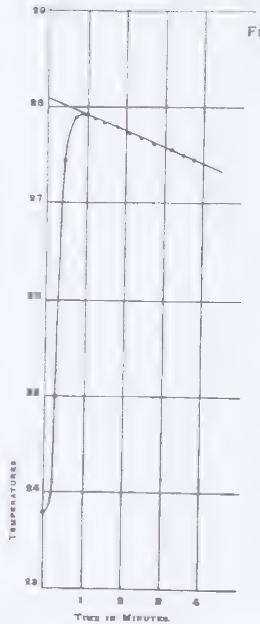
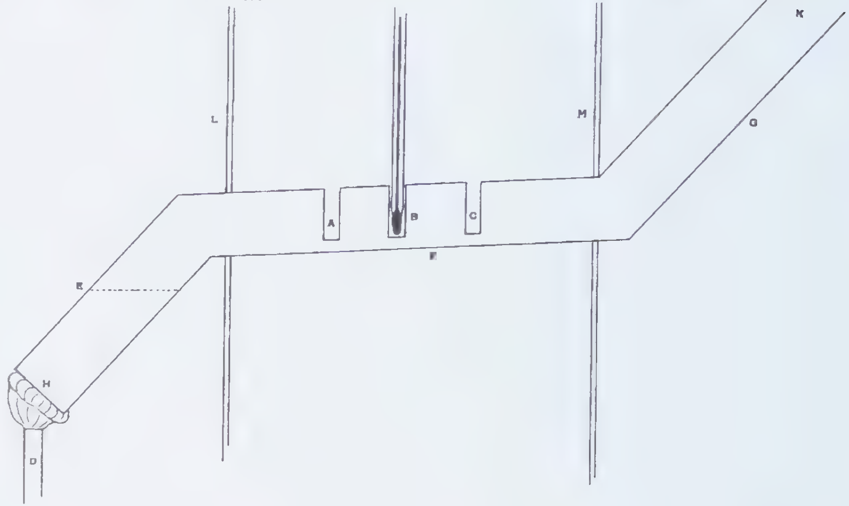


Fig. I.—Curve illustrating the method of applying the correction for cooling in specific heat determinations

Initial Temperature of Water 23·62°C.
Corrected Temperature of Mixture 28·08°C.

Fig. III.



XII.—*The Absolute Thermal Conductivity of Nickel.* By T. C. Baillie, M.A., B.Sc., Assistant Lecturer and Demonstrator in Physics, University College of North Wales, Bangor. (With a Plate.)

(Read 16th May 1898.)

§ 1. Introduction.—The experiments described in this paper were commenced with the view, not only of determining the absolute thermal conductivity of nickel, but also of comparing the results found by Forbes's and Ångstrom's methods for the same specimen. Although some readings were taken for Ångstrom's method, that part of the investigation was not completed, because it was found that the experimental errors — unavoidable, on account of the necessity of measuring rapidly changing temperatures—would be too great for the results to be of any value. The thermal conductivity of a portion of the bar of nickel used for Forbes's method was determined by a direct method involving the determination only of steady temperatures, and the results so obtained are given in the latter portion of this paper.

Forbes's Method.

§ 2. The Statical Experiment.—The nickel bar used was kindly lent by Dr Knott, being a piece about four feet long, which he had no immediate occasion to use for his own experiments on "The Strains produced in Iron, Steel, and Nickel Tubes in the Magnetic Field" (see *Trans.*, vol. xxxviii. part iii. No. 13). This bar was turned down so as to be of uniform circular section, and holes for thermometers were drilled into it by Messrs Aitken & Allan, Edinburgh. Four thermometer holes were drilled in each of the end portions of the bar, so as to leave a length of 19 inches intact for a tube required by Dr Knott at a later period.

The bar was set up in Professor Tait's private laboratory, with the same fittings, altered to suit the size of the nickel bar, as were used by Professor Tait, and afterwards by Dr Mitchell, in their experiments on thermal conductivities (see *Trans.*, 1878, xxx., and *Trans.*, 1887, xxxiii. part ii. p. 535). One end of the bar was fitted with white lead into a round hole in the side of a cast-iron pot, which was afterwards nearly filled with solder. This end of the bar was heated by a bunsen flame placed under the pot of solder. A constant temperature was maintained at this end of the bar by keeping the gas supplied to the bunsen burner at constant pressure by means of Professors Tait and Crum Brown's gas regulator. This regulator is like a small gasometer, one of the balancing weights of which, on descending, bends a piece of soft, flexible rubber tubing conducting the gas supply to it, so as to diminish the internal cross-section of the piece of soft tubing;

on the weight ascending, the cross-section of the tubing is increased. The nickel bar was protected by a double metal screen from thermal disturbances due to the heater. A constant stream of cold water was kept playing on the unheated end of the bar. The thermometers used were some of the Kew standard thermometers used by Professor TAIT and Dr MITCHELL. Temperatures above 200° C. were not used, and as the readings on the majority of the thermometers varied in the course of an experiment through a range of about 1° C., the thermometers were simply read by the naked eye to the nearest quarter of a degree. One could be quite certain of avoiding making any error due to parallax of more than that amount. A little mercury was put in each of the thermometer holes to give good thermal contact between the bulbs of the thermometers and the bar. No amalgamation of the nickel has ever been observed.

A steady state as regards the distribution of temperature along the bar was not reached in five hours from the time the gas at the heater was lighted, and the readings on the thermometers usually increased at the rate of about a quarter of a degree per hour for a few hours more. The following table shows the readings (uncorrected) obtained over an interval of nearly twenty-four hours on 18th July 1894 and the following morning. The gas was lighted at 8.30 A.M.

Time.	1	2	3	4	5	6	7	8	Temperature of Air.
12.30	175	130·5	100	78	26	23	20·5	18·2	19·0
1.30	177	133	102·5	80·5	28	25	22	19·1	19·1
2.0	177	133·3	103	81	28·9	25·4	22·5	19·5	19·1
2.15	176·5	133	103	81·5	29	25·8	22·5	19·7	19·1
2.45	176·5	133	103	81·5	29·4	26	23	20	19·1
6.0	177·8	134	104	82·2	30	26·5	23·2	20	19·1
6.30	178	134·3	104·5	82·5	30	26·5	23	20	19·2
7.0	178	134·3	104	82·5	30	26·5	23·1	20·1	19·2
7.30	177·5	134	104	82·3	30	26·5	23·2	20·1	19·4
8.0	178·5	134·5	104	82·5	30·1	26·7	23·4	20·2	19·3
8.30	178	134	104	82·2	30	26·6	23·1	20·0	19·3
9.0	178	134·3	104	82·3	30	26·5	23	20	19·4
9.30	178	134·3	104	82·5	30	26·5	23	20	19·4
10.0	178	134	104	82·5	30	26·5	23	20	19·3
11.20	179	135	105	83	30·1	26·6	23·1	20	19·4
1.30 a.m.	178	135	105	83	30·1	26·5	23·1	19·9	19·3
2.15	178	135	104·7	83	30·1	26·5	23·1	19·9	19·3
3.30	179·7	135·5	105	83·4	30·2	26·6	23	19·9	19·3
4.25	179	135·1	105	83·4	30·2	26·6	23	19·9	19·2
5.0	179	135	105	83·2	30·2	26·6	23	19·8	19·2
6.0	178·5	135	104·8	83	30	26·5	23	19·7	19·0

After several sets of readings had been taken, the bar was reversed, and heated at the other end. The thermometers were never shifted from their positions, except when the bar was reversed. They were read on each morning before the burner was lighted, and their readings on those occasions never differed by more than one-third of a degree. The thermometers were corrected for stem exposure by adding to the reading V. the

product ·000113 V^2. This correction seems to me to be probably too high for some of the thermometers, but it is less than that which Dr MITCHELL applied to the same thermometers, viz., ·00016 V^2. The value of the stem correction which I have chosen is based on the results of experiments to be described later in connection with the other method of determining the conductivity.

The dimensions, etc., of the bar were as follows:—

Length of bar, ...	128·5 cm.
Mass of bar, ...	19·28 kilo.
Diameter of bar, ...	4·67 cm.
Density of nickel, *i.e.*, mass ÷ volume, . . .	8·724 gms. per c.cm.
Specific gravity of a small piece cut off, determined by weighing in air and in water, . . .	8·75

Distances of thermometer holes from one end of the bar:—

Number of Hole.	Distance to Centre of Hole.
1	14·75 cm.
2	23·02 „
3	31·34 „
4	39·73 „
5	88·72 „
6	97·02 „
7	105·26 „
8	113·59 „

The statical experiment was frequently repeated at the same and different temperatures, and the following table contains the readings chosen for calculation corrected for stem exposure. On 6th August and the following days the bar was heated at the opposite end to that at which it was heated on previous occasions. The table shows that any effect due to tarnishing of the surface during the few weeks occupied by these experiments is not noticeable.

Date of Experiment.	Numbers of Thermometer Holes.								Temperature of Air.
	1	2	3	4	5	6	7	8	
July 12	150·5	114·8	90·2	72·3	28·9	26·0	23·2	20·6	20·2
„ 13	180·6	135·0	103·5	81·7	28·9	25·4	22·5	19·7	19·4
„ 16	181·7	135·6	104·4	82·1	29·3	26·0	22·8	20·0	18·8
„ 17	181·0	136·0	104·9	82·8	29·8	26·1	23·0	20·0	19·1
„ 18	181·6	136·0	105·2	83·2	30·1	26·5	23·0	20·1	19·3
„ 27	69·8	57·0	47·7	40·8	22·0	20·7	19·2	18·0	18·5
„ 30	116·7	100·1	72·8	59·9	26·5	24·1	22·0	20·0	18·9
„ 31	116·6	99·9	72·5	59·6	26·5	24·4	22·4	20·5	19·1
August 1	72·5	59·4	49·4	42·3	23·0	21·8	20·4	19·0	19·0
„ 2	67·5	55·4	46·8	40·3	22·7	21·5	20·4	19·1	18·6
„ 3	67·5	55·3	46·7	40·3	23·9	23·0	22·4	21·9	18·7
„ 6	150·3	114·2	89·7	71·3	28·5	25·7	23·1	20·9	18·9
„ 7	198·8	147·9	113·8	89·2	31·6	27·9	24·5	21·4	19·2
„ 8	109·3	85·9	69·5	57·0	25·9	23·7	21·6	19·5	18·9
„ 9	70·0	57·4	48·3	41·2	22·8	21·4	20·0	18·8	18·3
„ 10	165·3	124·9	97·6	77·6	29·8	26·6	23·8	21·1	18·9
„ 13	163·5	123·2	96·0	75·6	28·3	25·3	22·7	20·0	18·4

§ 3. Reduction of the Readings.—The equation for the conduction of heat in a bar, each part of which is at a steady temperature, is:—

$$KA\frac{d^2\theta}{dx^2}=Ep\theta,$$

where K is the thermal conductivity, A the cross-section, E the emissivity, and p the perimeter of a part of the bar, at temperature θ above the surrounding air, and at a distance x from some fixed point in the axis of the bar. Since K, A, E, and p are either constants or functions of θ only, it follows that $\frac{d^2\theta}{dx^2}$ is a function of θ only, and therefore the value of $d^2\theta/dx^2$ for any given value of θ should be the same, no matter which of the sets of readings it is derived from. This affords a means of testing the concordance of the various sets of readings. The determination of $d^2\theta/dx^2$ directly—by drawing a curve representing θ as a function of x, taking the tangents at various points, and thus getting another curve showing $d\theta/dx$ as a function of x; and from this, by a similar process, another showing $d^2\theta/dx^2$ as a function of x, and using the first and last curves to get $d^2\theta/dx^2$ as a function of θ—does not give $d^2\theta/dx^2$ with sufficient accuracy. A common proceeding is to find suitable values of the constants in some empirical equation representing θ as a function of x, and to differentiate the equation to obtain $d\theta/dx$. The following method of reducing the readings obtained in the statical experiment was adopted after trying others. Curves were made from the sets of readings on the first four thermometers only, in which log. θ was shown as a function of x. The gradient of these curves increased, but not very rapidly, with log. θ, and therefore $d(\log.\ \theta)/dx$ increased as θ increased. The curves were drawn by a lath, to the ends of which couples were applied so as to give it the slight curvature necessary to make the curve produced by its means pass in close proximity to each of the four points given by the corrected readings of the thermometers. It was noticed that the value of $d(\log.\ \theta)/dx$ was practically the same, for the same value of θ, for all curves. A new curve was then constructed, in which $d(\log.\ \theta)/dx$ was shown as a function of θ. The different points found on this curve lay very approximately in a straight line—that is to say, $\frac{d}{d\theta}\left(\frac{d(\log.\ \theta)}{dx}\right)$ was practically constant. The equation to the statical curves must then be of the form $\frac{1}{bc}\log.\frac{\theta}{\theta+b}=x+B$, where b and c have the same value for each curve. The simplicity of this method of finding the average values of $d^2\theta/dx^2$ for all sets of readings was what led to its adoption. In any case, $d(\log.\ \theta)/dx$ does not vary so rapidly as $d\theta/dx$, and it is therefore easier to get $d\theta/dx$ with accuracy, when using graphical methods, by multiplying $d(\log.\ \theta)/dx$ by θ, than it is to get $d\theta/dx$ directly. The values found for the constants in the above equation were $c=\cdot0000505$, and $b=670$. The value of $d^2\theta/dx^2$ is $c^2(2\theta+b)(\theta+b)\theta$. The following table contains the values of $d^2\theta/dx^2$ calculated, not from the expression just given, but from the

numbers found in the curve, using the formula

$$\frac{d^2\theta}{dx^2} = \theta\frac{d\,(\log.\,\theta)}{dx}\left\{\frac{d\,(\log.\,\theta)}{dx} + \theta\frac{d}{d\theta}\left(\frac{d\,(\log.\,\theta)}{dx}\right)\right\}.$$

The temperatures given in the tables are actual temperatures, not temperature excesses. They have been got by adding 19—about the average temperature of the air during the experiments—to the numbers used in the curves which were differences of temperature between the bar and the air.

Temperatures.	$\frac{d\,(\log.\,\theta)}{dx}$	$\frac{d^2\theta}{dx^2}$	Temperatures.	$\frac{d\,(\log.\,\theta)}{dx}$	$\frac{d^2\theta}{dx^2}$
40	·0335	·0239	110	·0370	·138
50	·0340	·0369	120	·0375	·159
60	·0345	·0509	130	·0380	·181
70	·0350	·0661	140	·0385	·205
80	·0355	·0823	150	·0390	·230
90	·0360	·0997	160	·0395	·256
100	·0365	·1183	170	·0400	·284
			180	·0405	·313
			190	·0410	·344
			200	·0415	·376

§ 4. THE COOLING EXPERIMENT.—For this experiment a short bar turned down from a left-over portion of Dr KNOTT's nickel bars was used. It was a piece of the same rod as the bar used in the statical experiment: it was turned down in the same way, at the same time, and to the same diameter, as was found by careful measurement. The length of the cooling bar was 21·55 cm., and as its diameter was 4·67 cm., the surface exposed at the ends was 9¾ per cent. of the whole surface. This involves an increase in the rates of cooling of about ten per cent. This is a serious drawback in these experiments. It has been allowed for by diminishing the observed rates of cooling in the ratio of the whole area of the cooling bar to the area of the curved portion only. It is possible that in air the emissivity of a vertical surface is, *ceteris paribus*, greater than the average emissivity of a curved cylindrical surface of the same diameter. As there is heat lost from the ends of the cooling bar, there must be some fall of temperature between the centre and the ends. I have given up all attempts at making allowance for this. The best way of meeting difficulties of that kind is to make the end correction negligible altogether. The bar was heated over a row of bunsen burners, without the previous warming necessary to avoid "sweating." The bar was heated pretty rapidly, and turned round rapidly while being heated, and very little moisture condensed upon it. A Kew mercury thermometer was used to measure the rate of cooling of the short bar which was heated to about 250° C. Readings were not taken until the bar had cooled for some time with the thermometer in position, since the distribution of temperature in the thermometer itself is at first irregular. This is discussed very fully by Professor TAIT in his paper already referred

to. The thermometer was observed through the telescope of a cathetometer, and the time at which the top of the mercury column passed each degree division mark was noted by looking at that instant at the dial of a watch. After some practice it was found easy to note the time of such transits to within a couple of seconds from the position of the seconds' hand, without paying much attention to the divisions round the dial. The time of transit set down for each degree division was late by the time taken to look from the thermometer to the watch, but as this is small and affects each reading, it is of no consequence. When the cooling became comparatively slow, as it did below 100° C., it was possible to see the top of the mercury column disappear behind a degree division mark, note the time, and have the eye in position at the telescope again in time to see the top of the column reappear on the under edge of the division mark.

§ 5. Reduction of the Cooling Readings.—The method employed for reducing these readings was as follows:—On paper ruled in squares temperatures were plotted as *abscissae*, and the times (in seconds) taken by the bar to cool through one degree were plotted as ordinates corresponding to the mean temperatures for those degrees: thus, for example, the ordinate corresponding to the temperature 102·5° C. was the observed time taken by the bar to cool from 103° C. to 102° C. The advantage of this method of reduction is the simplicity of correcting for errors of observation, &c. Suppose, for example, that the time set down for the transit across the 175 degree division mark is too late, the time noted for cooling from 176° to 175° is too great, and the amount by which it is unduly increased is deducted from the time of cooling from 175° to 174°; but the average time of cooling for a range including 176° to 174° is not affected by the supposed error. An error in graduation, by which one of the division marks is displaced, produces a similar effect. The ordinates would, if there were no errors of any kind, increase in length continuously as the temperature diminishes. If the curve formed by the ends of the ordinates is not continuous, all that is necessary is to make a continuous curve by reducing the lengths of those ordinates which are obviously too long and increasing the lengths of adjacent ordinates, and *vice versa*, so as to keep the sum total of the lengths of all the ordinates constant. This treatment will get rid of the effects of errors such as those considered above. The way in which this was carried out was to form a new curve in which for each reading was substituted the average of the five nearest readings. This gave a curve which was smooth but with small "waves" along it. A mean curve was then drawn by means of a lath planed thinner towards one end so as to produce the necessary variation of curvature along it. The ordinate at any temperature of the curve so constructed is the reciprocal of the rate of cooling at that temperature.

A cooling experiment was done alongside of the statical experiment on several days, until the surface of the cooling bar was thought to be just perceptibly dimmer than that of the long bar. It was confidently expected that the repeated heating of the short bar, especially as "sweating" was not entirely avoided, would affect the surface and increase its emissivity. The readings taken show that each time the bar was heated its emis-

sivity was increased before the tarnish on the surface became even perceptible. The following table will show this, and it provides the means of allowing for it.

Date of Experiment.	Time of Cooling from 200° to 150° in Seconds.	Temperature of Air.	Time of Cooling from 150° to 100° in Seconds.	Temperature of Air.	Time of Cooling from 100° to 70° in Seconds.	Temperature of Air.	Time of Cooling from 70° to 40° in Seconds.	Temperature of Air.
1894								
July 12	1375	20·4°	2210	20·3°	2295	20·3°	5125	20·2°
„ 13	1355	19·5°	2175	19·3°	2280	19·3°		
„ 17			2087	19·1°	2193	19·1°	4795	19·1°
„ 18			2105	19·1°				
August 2	1348	18·8°	2150	18·7°	2201	18·6°	4660	18·55°
„ 3					2191	18·7°	4503	18·7°
„ 7			2178	18·3°	2214	18·5°		
„ 8	1341	19·2°	2149	19·2°	2233	19·1°	4740	19·1°

The table also shows irregularities in the cooling, due possibly to changes in the state of the atmosphere, or to variations in the unavoidable draughts of the second order of magnitude. The readings obtained on 8th August gave the best curves, and they have been used to determine the rates of cooling given in a later table. A reduction of 2 per cent. was made in the rates of cooling found, in order to allow for the increase in the emissivity which had taken place by 8th August. A correction is necessary in the cooling experiment for the exposed stem of the thermometer. In the statical experiment the stem correction was made by adding to the observed reading V. the product $\cdot000113\ V^2$. Using the same form of correction in the cooling experiment, let V. be the observed, and θ the true temperature, then since $\theta = V. + \cdot000113\ V^2.$, the true rate of cooling $d\theta/dt$ is equal to $(1 + \cdot000226\ V.)$ times the apparent rate of cooling dv/dt got from the curves in which stem exposure is not allowed for. Since the thermometer parts with its heat to the cooling bar, its temperature must always during the cooling be higher than that of the bar; in other words, the thermometer lags behind the bar by an amount depending on the rate of cooling. No attempt has been made to allow for that in these experiments. Some mercury was put into the hole for the thermometer to give good thermal contact between the bar and the thermometer. The following table gives the rates of cooling of the short bar found after applying the corrections referred to above, except that due to lag, and that due to loss of heat at the ends of the bar.

Temperature.	Rate of Cooling.	Temperature.	Rate of Cooling.	Temperature.	Rate of Cooling.
40	·00362	90	·0149	150	·0308
50	·00556	100	·0174	160	·0336
60	·00770	110	·0200	170	·0363
70	·01005	120	·0225	180	·0392
80	·01250	130	·0251	190	·0423
		140	·0279	200	·0455

§ 6. Specific Heat of Nickel.—The determination of the specific heat of the nickel has been found by far the most troublesome part of these experiments. A portion of the cooling bar about 2·5 inches long had a hole drilled into it to receive the thermometer. A little mercury was put into the hole along with the thermometer. It was then heated and allowed to cool. At some instant the temperature was noted just as it was let fall into a large calorimeter, and the heat given out by the nickel was measured in the ordinary way by the method of mixtures. This was repeated at the same and different temperatures, and the results were not quite concordant, but indicated that the specific heat increased with temperature. As it was not quite certain that the temperature in all parts of the interior of the piece of nickel was that of the mixture when the readings were taken nickel turnings were tried. Several pieces of the turnings made in turning down the nickel bars were tied together with a short piece of thread, whose mass was negligible, and heated in the inner chamber of a double cylinder of copper containing glycerine between the cylinders. This was heated to over 200°C. and packed up with cotton wool in a wooden case provided with a contrivance for opening a slide at the bottom and allowing the nickel to fall into the calorimeter at the moment of opening. The calorimeter used was a small glass beaker of suitable dimensions. With it the cooling correction was smaller than with a copper calorimeter of the same size. It was hoped that as the heater cooled very slowly it would be safe to assume that the temperature of the turnings after being in the heater some time would be that of the thermometer whose bulb was inserted amongst them. As the heater cooled, determinations of the specific heat could be done on the same day at lower and lower temperatures. It was found that the sets of determinations obtained on separate days did not agree no matter how long the nickel was kept in the heater, and as all the quantities involved could be measured within 1 per cent., and the correction for cooling was only about 1 per cent., the heater was regarded as the cause of the irregularities.

In some subsequent experiments the nickel turnings were heated in a steam jacket of the usual laboratory pattern, and results agreeing within 2 per cent. were obtained when the nickel was in the heater for not less than two hours. As the heat required to raise the temperature of 1 gramme of water is not constant, but varies in a manner depending on the thermometer used in the calorimeter, closer agreement than this is not to be expected. The specific heat thus obtained was higher than that got for the same temperature from the large mass cut off the cooling bar. At the same time there is *no* reason for supposing that the specific heat would *not* be affected by the nickel being cut up and distorted as it is in the form of turnings.

The values of the specific heat given below were found from a solid piece of the nickel weighing nearly 100 grammes, and as a glass calorimeter could not be used with so large a mass, a copper calorimeter was made of thin sheet copper, the depth of it being 5 inches, and the diameter $2\frac{1}{4}$ inches. There was a slight recess along one side to accommodate the thermometer and a flange round the lip by which it was suspended

in the interior of a large copper vessel which protected it from draughts in the room. The thermometer used in the calorimeter was one of DUCRETET's précision thermometers divided into tenths of degrees centigrade, and it was read by means of a telescope fixed a little distance off, hundredths of a degree being estimated by eye. The steam heater was used in the ordinary way, but in order to get determinations at different temperatures the same heater was used with methylated spirits instead of water. A tube was arranged to conduct all the spirits which condensed in the apparatus back to the boiler by a pipe leading in at the bottom of it. Only a small portion of the spirits was distilled off, as the flame of the burner heating the boiler was so arranged that very little vapour was formed over and above that required to produce heat enough by its condensation to maintain the heater at a uniform temperature. The boiling point rose by less than one degree in the course of a day on account of loss by distillation of the more volatile constituents of the spirits. The top of the chamber in the heater, in which the nickel was suspended while being heated, was closed by a large cork in which were two holes, one letting in the thermometer which indicated the temperature of the nickel, and another through which passed a fine wire supporting the nickel. The bottom of the chamber was closed by a slide padded with cotton wool. This slide was drawn aside when the nickel was dropped into the calorimeter, an arrangement being made for doing all this with great rapidity. The wire which had supported the nickel in the heater remained attached to it, and the end of it, which was usually found projecting out of the mouth of the calorimeter, was at once seized and the contents of the calorimeter stirred by moving the nickel up and down and to and fro in the water. While this was being done the thermometer was being watched through the telescope. Two persons were thus necessary. No correction has been made for the thermal equivalent of the work done in stirring, as it has been assumed to be negligible. It was impossible to observe if stirring the water produced an appreciable quantity of heat as under all circumstances its effect was quite obscured by the disturbances produced by other causes.

The correction for cooling was applied in the following way:—The observer at the telescope had in his hands a stop watch, with two hands so arranged that, by pressing one stop, they would start together; pressing another stop made one of them remain where it was at the instant of pressing; a second press made it overtake and go on as before with the other hand. In this way, the time at which the thermometer indicated any reading could be noted to a fraction of a second, the reading being written down subsequently; and a fresh reading could be then taken in the same way. A curve was then plotted from the readings thus obtained, with temperatures as ordinates, and times as *abscissae*. The part of the curve corresponding to times after the maximum temperature had been reached was produced backwards to the axis of zero time, and in this way the temperature which the calorimeter must have cooled from, had its rise of temperature been instantaneous, was found. This is very nearly the temperature which would have been reached if the calorimeter had not lost any heat at all. This correction is probably

too much by something less than one-half per cent. An example of such a curve of correction for cooling is given in fig. 1.

At temperatures over 100° C. another form of heater was used. An iron tube was surrounded by a conical-shaped iron chamber riveted on to it, and mercury was put in the space between. This was heated by a circular gas burner, the flame of which was regulated by the volume of the mercury, which, on expanding above a certain limit, cut off all the gas, except what found its way through a small by-pass. The by-pass was arranged to allow just sufficient gas through it to keep the burner lighted, and thus to save the trouble of lighting the gas when the mercury had contracted sufficiently. The arrangement was similar to that shown at E in fig. 2. The flame rose and fell about ten times in a minute. The temperature at any one place in the heater was very steady after it had been in action for an hour or two, but the temperature near the top of the inner tube was 2 or 3 degrees lower than that of the hottest part. The thermometer used for reading the temperature of the nickel in this heater was put in so that the bulb touched the nickel. The nickel dropped into the calorimeter through the centre of the ring burner. The inner tube of the heater was prolonged below the burner. Corrections for stem exposure have been applied to the readings of the thermometers. It was possible to obtain only an approximation to the stem corrections on account of the manner in which the thermometers were placed, with some part of their stem at unknown temperatures; but as the correction is only 2 per cent. in the greatest instance, they are probably accurate enough.

The following tables give the data obtained in the last sets of experiments done :—

Set I.

Mass of nickel, ...	99·3 grammes.
Mass of water in calorimeter, . . .	156·7 "
Water equivalent of calorimeter, etc., . .	3·5

Date of Experiment.	Initial Temp. of Water.	Corrected Temp. of Mixture.	Rise of Temp. of Water.	Initial Temp. of Nickel.	Fall of Temp. of Nickel.	Log. Ratio Rise of Temp. of Water to Fall of Temp. of Nickel.
1897 July 7	16·94	21·87	4·93	99·1	77·2	$\bar{2}$·8052
" "	18·80	23·65	4·85	99·2	75·2	$\bar{2}$·8072
" "	19·98	24·83	4·85	99·3	74·5	$\bar{2}$·8135
" "	20·88	25·72	4·84	99·4	73·7	$\bar{2}$·8173
" 8	18·65	23·54	4·89	99·4	75·9	$\bar{2}$·8091
" "	19·60	24·46	4·86	99·4	74·9	$\bar{2}$·8121
" "	19·12	23·99	4·87	99·4	75·4	$\bar{2}$·8101
Arithmetic mean of observed values	19·14	24·01	4·87	99·3	75·3	$\bar{2}$·8107

Average value of specific heat, ·104.

SET II.

Mass of nickel, ... 99·3 grammes.
Mass of water in calorimeter, ... 126·7 ,,
Water equivalent of calorimeter, etc., . . 3·3

Date of Experiment.	Initial Temp. of Water.	Corrected Temp. of Mixture.	Rise of Temp. of Water.	Initial Temp. of Nickel.	Fall of Temp. of Nickel.	Log. Ratio Rise of Temp. of Water to Fall of Temp. of Nickel.
1897						
July 9	17·10	23·13	6·03	99·4	76·3	$\bar{2}$·8978
,, ,,	18·10	24·08	5·98	99·4	75·3	$\bar{2}$·8999
,, ,,	19·40	25·34	5·94	99·4	74·1	$\bar{2}$·9040
,, ,,	19·88	25·72	5·84	99·5	73·8	$\bar{2}$·8983
,, 10	18·19	24·28	6·09	99·6	75·3	$\bar{2}$·9078
,, ,,	20·25	26·12	5·87	98·6	72·5	$\bar{2}$·9083
Arithmetic mean of observed values	18·82	24·78	5·96	99·3	74·55	$\bar{2}$·9027

Average value of specific heat of nickel, ·105.

SET III.

Mass of nickel, ... 99·3 grammes.
Mass of water in calorimeter, ... 126·7 ,,
Water equivalent of calorimeter, etc., . . 3·3

Date of Experiment.	Initial Temp. of Water.	Corrected Temp. of Mixture.	Rise of Temp. of Water.	Initial Temp. of Nickel.	Fall of Temp. of Nickel.	Log. Ratio Rise of Temp. of Water to Fall of Temp. of Nickel.
1897						
July 13	21·05	25·27	4·22	77·3	52·0	$\bar{2}$·9093
,, ,,	22·27	26·31	4·04	77·5	51·2	$\bar{2}$·8971
,, ,,	23·80	27·82	4·02	78·1	50·3	$\bar{2}$·9026
,, 14	21·01	25·23	4·22	78·2	53·0	$\bar{2}$·9010
,, ,,	22·13	26·20	4·07	77·5	51·3	$\bar{2}$·8995
,, ,,	23·55	27·51	3·96	77·5	50·0	$\bar{2}$·8987
Arithmetic mean	22·30	26·39	4·09	77·7	51·3	$\bar{2}$·9015

Average value of specific heat of nickel, ·105.

SET IV.

Mass of nickel, ... 99·3 grammes.
Mass of water in calorimeter, ... 156·7 ,,
Water equivalent of calorimeter, etc., . . 3·5

Date of Experiment.	Initial Temp. of Water.	Corrected Temp. of Mixture.	Rise of Temp. of Water.	Initial Temp. of Nickel.	Fall of Temp. of Nickel.	Log. Ratio Rise of Temp. of Water to Fall of Temp. of Nickel.
1897						
July 20	21·60	29·75	8·15	145·7	116·0	$\bar{2}$·8467
" "	22·80	30·93	8·13	146·4	115·5	$\bar{2}$·8475
" 21	20·31	28·73	8·42	147·3	118·6	$\bar{2}$·8511
" "	21·12	29·32	8·20	147·3	118·0	$\bar{2}$·8419
" "	21·90	30·08	8·18	147·3	117·2	$\bar{2}$·8438
" 22	18·78	27·35	8·57	148·1	120·8	$\bar{2}$·8510
" "	18·99	27·50	8·51	148·8	121·3	$\bar{2}$·8461
" "	19·63	28·05	8·42	149·1	121·0	$\bar{2}$·8425
" 23	21·06	29·43	8·37	149·0	119·6	$\bar{2}$·8449
Arithmetic mean	20·69	29·02	8·33	147·67	118·65	$\bar{2}$·8462

Average value of specific heat of nickel, ·113.

The average value of the specific heat of the nickel turnings for a range varying from just under 100° C. to about 20° C. was about ·11. This shows that either the thermal capacity is altered in the process of disintegration, or that there is some error in the determination depending upon the size of the pieces employed. The latter I believe to be the case. During the month of June, I did several determinations at the same time with a bundle of copper washers, and with the piece of nickel referred to. After a few trials I found the mass of copper (114·8 grammes) which had the same thermal capacity as the 99·3 grammes of nickel. I tried to discover a difference between the rate of rise of temperature in the calorimeter when the copper was employed from that when the nickel was used. The difference in the times taken to reach the maximum reading was only about six seconds, the whole time being about one minute to one and a quarter. Probably the lag of the thermometer behind the calorimeter obscured the greater part of the actual difference.

The effect of not receiving all the heat from the nickel would be to make the apparent specific heat less than the true specific heat. This error would obviously be greater at low temperatures than at high temperatures, and thus would make the apparent specific heat increase more rapidly with temperature than the true specific heat actually does. Probably this is the real reason why the specific heats of carbon and silicon—so-called bad conductors of heat—have been found to be much lower at ordinary temperatures than that expected from DULONG and PETIT's law of constant atomic heats, whereas at very high temperatures their specific heats are much greater and nearly great enough to fulfil the law. Errors of this kind are reduced to a minimum by using WATERMAN's calorimeter. A description of this apparatus, and a short discussion of the determinations of specific heats is given in a paper by WATERMAN in the *Physical Review* (vol. iv. No. 3) for December 1896.

§ 7. It seems to me a disadvantage of FORBES's method that its accuracy has to

depend on that of the determination of specific heat. While I have no confidence in the values found for the specific heat of the nickel, I give the values of the conductivity found by using them. I hope to be able at a future time to supplant these figures by others which can be relied on.

The following table gives the values of the ratio of the conductivity to the specific heat after applying the end correction to the rates of cooling given in a previous table, and the values of the conductivity using the values of the specific heat in the adjacent column. No corrections have been applied for changes of the dimensions of the nickel with temperature as these are really negligible.

Temperature.	Ratio of Conductivity to Specific Heat.	Specific Heat.	Conductivity.
40	1·19	·098	·118
50	1·19	·102	·121
60	1·19	·105	·125
70	1·20	·108	·130
80	1·20	·111	·133
90	1·18	·114	·135
100	1·16	·118	·137
110	1·14	·121	·138
120	1·12	·124	·139
130	1·09	·127	·139
140	1·07	·130	·140
150	1·06	·134	·142
160	1·04		
170	1·01		
180	·99		
190	·97		
200	·96		

§ 8. Experimental and other Errors in Forbes's Method.—The sources of error may be classified as follows :—

Statical Experiment.

(1) Thermometric errors.

(2) Errors in reduction of results, for example in differentiating the temperature curve.

(3) Want of uniformity or regularity in the substance or surface of the Forbes bar.

Cooling Experiment.

(4) Radiation from ends of bar.

(5) Lag of thermometer behind bar due to gradient of temperature necessary to cause flow of heat from thermometer to bar.

(6) Thermometric errors.

(7) Errors of observation in taking cooling readings.

(8) Errors in reduction of rate of cooling from these readings.

(9) Difference in the emissive powers of the surfaces of the cooling bar and statical bar. Some of the causes of such differences may be—

(*a*) tarnish.

(*b*) differences in the amount of polish.

(*c*) difference in the surroundings or in the state of the atmosphere during the cooling and statical experiments.

(*d*) differences in the radiation due to the temperature of the cooling bar always falling while that of the statical bar is steady.

(10) Errors in the determination of the specific heat.

Of these the chief are—

(*a*) the specimen used for this may not be a fair average specimen.

(*b*) want of uniformity in its temperature when put into the calorimeter.

(*c*) the calorimeter not receiving the whole of the heat supposed to be given out from the specimen.

(*d*) changes in the thermal capacity of water with temperature as measured by the thermometer used (or, in the case of ice calorimeters, errors in the value of the latent heat or other constants used).

Of these there is little difficulty in arranging the errors from (1), (2), (4), (6), (7), (8) to be small by using proper care and suitably arranging the apparatus or the bars used. Serious error from (3) could be detected by taking a sufficient number of properly varied sets of readings. Small errors due to (9) (*b*) and (9) (*c*) are difficult to avoid and it is impossible to discover their existence. (10) (*c*) is the most serious cause of error in the ordinary method of mixtures. There is probably some error from this cause even in BUNSEN's calorimeter, as it usually gives lower values than other methods. (10) (*d*) is unavoidable in all thermometric thermal measurements. (5) and (9) (*d*) are inherent to the method and are not to be avoided by the use of thermo-electric junctions instead of thermometers. It is also impossible to estimate the errors arising therefrom. In ÅNGSTROM's method errors from (10) affect the result in the same way, and as all the temperatures measured are varying temperatures, errors of the same sort as (5) and (9) (*d*) may occur. ANGSTROM's method is unreliable on other grounds. It is essentially based on the assumption that the ratio of the conductivity to the emissivity is constant.

The values of the conductivity of copper found by Professor TAIT were (reduced to C.G.S. units) for good conducting copper $1{\cdot}08\ (1 + {\cdot}0013t)$; for bad conducting copper ${\cdot}71\ (1 + {\cdot}0014t)$. The ratio of these values is independent of nearly every source of error mentioned, and yet Dr STEWART (vide *Trans. Roy. Soc.*, 1893, p. 569) found $1{\cdot}12\ (1 - {\cdot}001t)$; while KIRCHHOFF and HANSEMANN (*vide* WIEDEMANN's *Annalen*, 9, p. 1; 13, p. 406) found ${\cdot}51\ (1 + {\cdot}0057t)$ both for pure copper. One has doubts about believing that the wide range of variations of these values is due only to differences in the specimens of metal used. I, therefore, determined to find the conductivity of the nickel I had used by a method with fewer sources of error.

Direct Method.

§ 9. The Apparatus.—The method of determining thermal conductivity by direct measurement of the rate of flow of heat and gradient of temperature is that adopted in the following experiments. This method was used long ago by Clement and by Péclet (vide *Ann. de Chimie et de Physique*, 3e tom. ii. p. 107, 1841), and in their hands did not yield satisfactory results as they did not measure the temperatures of the metal itself, but it has been used with success by E. H. Hall, who utilised the metal experimented upon as one of a thermo-electric couple to measure its own gradient of temperature (vide *Pro. American Academy*, vol. xxxi. p. 271).

In the present investigation one end of the nickel bar used for Forbes's method was cut off, and an extra thermometer hole was drilled into it. Its surface was repolished. The dimensions were as follows :—

Diameter, from 4·660 to 4·667 cm.
Length, 42·55 cm.
Density, 8·75 grammes per c.cm.

No. of Thermometer Hole.	Distance in Centimetres from end at which the Rate of Flow of Heat was measured.
1	2·84
2	11·20
3	19·54
4	27·88
5	36·17

A shorter length would have sufficed, and it would have been an advantage to have made more thermometer holes. The bar was fitted up so that one end could be kept at any constant high temperature, while a flow of water could be kept cooling the other, the rise of temperature of the water and the mass of water passing per unit of time being measured. These data were sufficient to measure the rate at which heat left the end of the bar. The gradient of temperature at any point is given by the tangent to the curve drawn from the readings given by the thermometers.

A slide bench was erected in front of the table carrying the apparatus, and was arranged to carry a telescope which could be raised or lowered in a vertical line, and at the same time moved to and fro along the bench which was placed parallel to the axis of the nickel bar. The thermometers used in the bar were some of Professor Tait's Kew thermometers from the same stock as those used in the Forbes bar, and they were placed so as to hang vertically, this being tested by a plumb line. As the telescope could only move so as to be always horizontal, parallax was avoided. When the telescope was adjusted so that one of the thermometers was in focus, all were in focus for that same adjustment, which was never altered. The readings were estimated by eye to the nearest tenth of a degree.

A diagram representing a vertical section through the axis of the bar is given in fig. 2. The heater was the cast-iron pot, J, which was used with the Forbes bar. The bar was fixed into the circular hole (at K) in the side of it with red lead. A Jena glass flask,

F, with a fairly long neck was filled to the bottom of the neck with mercury and put into the pot; and the space, H, between was filled nearly full of mercury. To prevent mercury leaking through the cast-iron pot it was previously lined with pipe-clay, a paste of pipe-clay and water being painted in with a brush and allowed to dry. Into the neck of the flask was fitted a glass piece made as shown in fig. 2. The arrows show the path taken by the gas to reach the burner, and the temperature was kept constant by the mercury cutting off the gas supply at E on reaching a certain temperature. The by-pass, B, was opened to allow a full supply of gas while the heater was being warmed up to the proper temperature, the mercury in the flask being allowed to run over at D, which was at all other times closed. When the desired temperature was reached the by-pass, B, was nearly closed, enough gas being allowed to pass through it to keep the Argand burner, G, from going out. This gas regulator worked so well that a thermometer hung in the same place in the pot of mercury showed no variation exceeding one-tenth of a degree centigrade during a whole day.

At first a large steel cap was fitted on the end of the bar, with mercury inside it, the idea being to make it at once the heater and the regulator. It showed a steady, slow rise of temperature, and, although there was no visible leakage, in a few days fine drops of mercury were seen on the iron tray placed under the burner to catch the mercury in case of accident. No leakage of mercury could be noticed from it even under greater pressure from the inside while standing cold, and therefore the mercury must have leaked through pores too small to be noticed while the flame played upon them. The variation of the temperature of cut-off is a very delicate test of such leakage.

The rate at which heat was given out at the other end of the bar was obtained by measuring the rise of temperature and the rate of flow of the stream of water which played on the end of the bar. A brass cap, M, was fitted on the end of the bar, the water entering the space between it and the bar having its temperature measured at O, and the temperature of the water leaving the bar was measured at P. The thermometers at O and P were Anschutz thermometers graduated in fifths of a degree centigrade. The rate of flow of the water was found by observing the time taken to fill the flask, Q, of known capacity to the fiducial mark. The water was supplied at constant level from a chamber, S, containing the well-known inverted bottle device, R. Distilled water was used, but great difficulty was found in keeping the rate of flow regular until the plan was tried of making the outlet of a piece of glass tubing drawn out fine and broken off at the capillary portion. With this improvement the flow was very uniform, and the temperature of the water (at O) reaching the bar was also very steady, but the temperature of the water leaving the bar (at P) varied. When the water had been once used it was cooled by being put in the inner chamber of a double copper tank, while cold tap water was circulating in the outer chamber surrounding it. The same water was thus used over and over again.

The order of taking readings was as follows:—1°, the thermometers in the bar; 2°,

the temperature of the cold water (at O) going to the bar; 3°, the time at which the empty flask, Q, was put to catch the overflowing water; 4°, the temperature of the water leaving the bar (at P) was read every half-minute while the flask was filling; 5°, the time the flask was exactly filled to the fiducial mark; 6°, the temperature of the water entering the cap at O; 7°, the thermometers in the bar. All these readings varied little in the course of one evening, and the rate at which heat was given out at the end of the bar varied within 2 per cent. The following table gives the readings (uncorrected) taken on 6th January 1898.

Average Temperature of Water.	Average Rate of Flow of Heat in Calories per Second.	Temperature of Air.	Temperatures of Holes in Bar.				
			No. 1.	No. 2.	No. 3.	No. 4.	No. 5.
21·7	4·85	13·2	39·6	61·65	88·35	122·2	168·2
		13·2	39·6	61·7	88·4	122·4	168·1
21·8	4·86	13·2	39·6	61·7	88·4	122·4	168·1
		13·3	39·8	61·8	88·5	122·45	168·2
21·85	4·88	13·3	39·65	61·9	88·55	122·5	168·25
		13·3	39·65	61·8	88·5	122·55	168·45
21·86	4·83	13·3	39·65	61·8	88·5	122·55	168·45
		13·3	39·7	61·85	88·6	122·6	168·35
Mean 21·8	4·86	13·3	39·65	61·8	88·5	122·45	168·25

§ 10. Correction of the Thermometers.—The corrections of the thermometers were not found in the ordinary way, as there is always more or less doubt attached to any allowance that may be made for stem exposure on account of the impossibility of knowing the exact distribution of temperature along the stem of the thermometer. The method adopted was simple and allowed the testing to be carried out with the thermometers in as nearly as possible the same circumstances as they are in during the experiments.

The thermometers were tested at three different temperatures, at 0° C., about 100° C., and about 218° C. At 0° C., the correction was found by hanging the thermometers in a vertical position with their bulbs, and as much of their stem as was under the surface of the bar, embedded in powdered ice washed with distilled water. At the other two temperatures the apparatus, a vertical section through the centre of which is shown in fig. 8, was used. A piece of brass tubing of nearly the same diameter as the bar of nickel was cut into three lengths, E, F, and G, and these were brazed together as shown. The end, H, was closed, and three small tubes, A, B, and C, were brazed in the middle piece F. These tubes were about half full of Wood's alloy. The piece E was half-filled with water which was heated by the burner D, which was adjusted until just a small quantity of water vapour escaped at the open end K. The tubes, A, B, and C, were of about the same depth as the thermometer holes in the bar, and during the test the thermometers were suspended in a vertical position with their bulbs near the bottom of the tubes. Asbestos screens were fitted up at L and M to shield the thermometers

from the disturbing effects of the burner on the one side and the escaping vapour on the other. The arrangement is just that of a modified reflux condenser.

The thermometers were suspended with their bulbs in the mercury of the heater (at H in fig. 2), and the temperature of the heater was gradually raised to about 100° C., when the regulator was adjusted to act. The thermometers were left there under these conditions from morning till evening. As readings were always taken in the evenings, while the heater was set working in the mornings, the thermometers were never read until they had been at the same temperature for several hours. It was therefore thought necessary to keep the thermometers the same length of time at 100° C. before testing them at that temperature, so as to allow the glass to take on the same set that it had in the bar at the same temperature. It is possible that if an ordinary mercury thermometer is kept for hours at some temperature before it is read, its reading at the same temperature on some other occasion will only be the same after it has remained at that temperature for some hours.

The burner D was lit, and after the testing apparatus had been at 100° C. for some time, one of the thermometers was taken out of the mercury heater and quickly put into tube A. After the first two or three occasions, it was found easy to do this so dexterously that the reading on the thermometer did not fall more than 2° in the interval. After it had been in A for some time, during which the reading was constant, it was rapidly transferred to B, and by and by to C. The thermometers were hung up vertically by means of a plumb line, and the readings taken with the telescope. It was found that when E was too full of water, even when it was just over half-full, the readings in A, B, and C were not alike. When that was the case, the thermometer was left in one of the tubes until enough of the water had evaporated. The barometer was read sometime during the test and the true temperature of the bulb of the thermometer found from REGNAULT's tables. The difference between the observed reading on the thermometer and the temperature of the water vapour gave the whole correction at that temperature, the graduation correction and the stem exposure correction being thus lumped together. The same thing was gone through for each of the thermometers.

The same sort of process was repeated with the same apparatus, after the water had been dried out and naphthalene put in its place. Pure naphthalene was used, and as the boiling point of pure naphthalene has been determined on the air thermometer scale by CRAFTS, and has been found to be very constant, it is as satisfactory a "fixed" point on the scale of temperatures as one can wish for. The total correction of each of the thermometers was thus found at the temperature of the boiling point of naphthalene. The graduation corrections on the Kew thermometers used were known to be small, and hence it was only to be expected that an expression of the form $a + bt^2$ would represent the correction. This expression suited the values of the corrections found for all the thermometers except one to within a fifth of a degree, but the value of b was not the same in all cases, as it varied from ·00008 to ·000115. Curiously enough, b was smaller for those thermometers graduated up to 300° C. than for those which could not read

above 220° C. The corrections at 0° C. were zero for most of the thermometers. One read ·55° too low, but that was due to a small particle of the mercury having been shaken up into the top of the stem—probably during transit—from which it could not be again dislodged. It was on the strength of these results that $\cdot 000113t^2$ was used to give the stem correction in the FORBES bar experiments.

The following table gives the corrected mean readings obtained from the last three experiments, together with the values of the conductivity calculated from them.

Date of Experiment.	Temp. of Air.	Corrected Mean Temperatures of Holes in Bar.					Temp. at End of Bar.	Gradient at End of Bar.	Mean Temp. of Water.	Flow of Heat in Calories per Second.	Conductivity at End of Bar.
		No. 1.	No. 2.	No. 3.	No. 4.	No. 5.					
31/12/97	14·8	39·3	74·1	115·9	168·1	242·7	28·5	3·66	16·8	8·12	·130
4/1/98	13·2	47·0	79·0	118·9	170·0	243·2	37·7	3·23	22·1	7·225	·131
6/1/98	13·3	40·5	62·3	89·4	123·9	170·3	34·3	2·08	21·8	4·86	·136

§ 11. THEORY OF THE METHOD.—Let K be the conductivity, θ the temperature, X the distance from some fixed point on the axis of the bar, of a section of the bar of area A, across which H units of heat pass in unit of time, then

$$KA\frac{d\theta}{dx}=H.$$

Corresponding values of θ, H, and $\frac{d\theta}{dx}$ are given in the above table for the end section of the bar whose cross-section is 17·1 square centimetres (diameter is 4·663 cms.). The values of H given are subject to two corrections: (1) a correction for heat lost by radiation from the brass cap; (2) correction for the changes in the thermal capacity of unit mass of water with temperature. An estimate of the former error shows that it never exceeded 1 per cent., so that it is probable that these corrections combined do not exceed 2 per cent. They are rather smaller than the corresponding corrections in a specific heat determination.

The values of $\frac{d\theta}{dx}$ are liable to error from two sources: (1) thermometric errors in the temperature of the nearest thermometer hole; (2) arithmetical or geometrical errors in differentiating the temperature curve. Errors from both of these causes would have been reduced by having more thermometer holes, and what discordance there is between the values of the conductivity found from the three sets of readings given above is probably mostly due to errors in estimating $d\theta/dx$. Differences amounting to 2 or 3 per cent. are only to be expected. All these sources of error effect FORBES's method,—and, of course, also ÅNGSTROM's—but to these are added in FORBES's method all those arising from the cooling experiment.

The measurements referred to only determine the conductivity at temperatures somewhat above that of the air, but the conductivity could be found in a similar manner at other temperatures (such as slightly over 100° C., by allowing the water in the cap to be

evaporated into steam). Also, by using an electrical heater, the heat supplied at the hot end (subject to corrections for radiation) could be measured and the gradient of temperature at that end. Such experiments, however, were not carried out in this case, because it was seen, in the manner described below, that the conductivity varied little with temperature.

§ 12. Change of Conductivity with Temperature.—Before the brass cap was fitted on the end of the bar for the experiments just described readings were taken with the bar losing heat only by radiation. After the distribution of temperature became steady, the heat which passed any cross-section of the bar was lost by radiation from the rest of the bar beyond. The following table gives the temperatures obtained, thermometric corrections being applied.

Temperature of Air.	Corrected Mean Temperatures of Holes in Bar.				
	No. 1.	No. 2.	No. 3.	No. 4.	No. 5.
13·6	63·85	68·9	78·7	95·35	119·5
8·4	83·55	91·0	107·35	134·15	174·4
14·3	102·7	112·75	132·8	167·35	219·1
9·2	110·15	121·8	146·2	185·4	248·3

Curves were drawn from these readings, and differentiated. By supposing the bar prolonged by an amount slightly over the length of the radius, and producing the temperature curve to that point, one obtained the curve which would suit the bar if no heat had been lost from the end, at which place $d\theta/dx$ would then be zero. From the first set of readings the value of $d\theta/dx$ at the section which had the temperature 120·65° C. was found to be 3·55, and its distance from the point at which $d\theta/dx$ vanished was 38 centimetres. The average excess of the temperature of those 38 centimetres of the bar over the temperature of the surrounding air was 67·45°. This gives the following relation :—

$$KA \times 3{\cdot}55 = Ep \times 67{\cdot}45 \times 38\,,$$

where K is the conductivity at 120·65° C. and E the average emissivity under the conditions referred to. From the set of readings obtained on 31st December 1897, and given on page 19, the gradient at 120·65° C. was found to be 5·66, the gradient at 63·2° C. to be 4·33 ; and the distance between the points at these two temperatures was 11·72 centimetres. The average excess of the temperature of those 11·72 centimetres of the bar over the temperature of the surrounding air was 76·0°. If we *assume* the average emissivity to be the *same* in these two cases, we find that 1·23 is the value of that part of the gradient which is required to account for the heat lost by cooling over the 11·72 centimetres in the latter instance. For if

$$KA \times 3{\cdot}55 = Ep \times 67{\cdot}45 \times 38\,,$$

then

$$KA \times 1{\cdot}23 = Ep \times 76{\cdot}0 \times 11{\cdot}72\,.$$

If we deduct 1·23 from the gradient, 5·66, at 120·65° C. in the latter experiment (date 31st December 1897), we find the gradient (the remaining 4·43) which would cause the same heat to pass the cross-section at 120·65° C. as passes the cross-section at 63·2° C. with its gradient at 4·33. In other words 4·43 and 4·33 would be corresponding values of the gradients at 120·65° and 63·2° respectively if no heat were lost by radiation from the bar. The conductivities at these two temperatures are inversely as these numbers. This shows a diminution of conductivity of 2½ per cent., with a rise in temperature of about 60°. This is within the limits of experimental error. The assumption that the average emissivities for temperature excesses of 67·45° and 76° are the same is not likely to be correct. The emissivity in the latter case will be greater, probably by something of the order of 2 per cent. The effect of the increase of emissivity with temperature will be to reduce the apparent diminution of conductivity with rise of temperature, and might even change it into an increase, but in any case it would be very small and within the limits of experimental error.

The following tables give two sets of data obtained from the curves drawn from the corrected readings already given in tabular form.

Corresponding Values of			Average Temperature Excess.	Corresponding Values at another Section of Bar of		
x	θ	$d\theta/dx$		x	θ	$d\theta/dx$
0	63·2	0	67·45	38·0	120·65	3·55
8·73	63·2	4·33	76·0	20·45	120·65	5·66
0	108·5	0	125·0	30·0	188·7	6·00
18·23	108·5	5·335	130·8	30·55	188·7	8·13

From these are deduced the following :—

	θ	$d\theta/dx$
Corresponding values of θ and $d\theta/dx$ which would be found if no heat were lost from surface of bar,	63·2 120·6	4·33 4·43
Ditto,	108·5 188·7	5·335 5·55

These figures indicate a diminution of conductivity of the amount ·000066 per rise of temperature of 1° C. The conductivity cannot fall so much as this, and in any case the change of conductivity with temperature is within the limits of error of such experiments up to a temperature of 200° C.

§ 13. Conclusion.—The conductivity of nickel found by the direct method is ·132. There is some doubt about the third figure after the decimal point, and that figure is the only one affected by changes of temperature up to 200°C. It is interesting to note that

the value of the specific heat of nickel found by using nickel turnings, viz., ·11, would, if multiplied by the ratio of the conductivity to the specific heat at the mean temperature 60°, give a result in exact agreement with the above. It should, however, be stated that the specimen of nickel showed slight fissures. These were not serious enough to affect the readings sufficiently to make it noticeable in the appearance of the temperature curves, and the readings obtained from the FORBES bar do not show irregularities from such a cause. The nickel used was also very pure. I am much indebted to my colleague, Mr F. V. DUTTON, for analysing it for me with the following result :—

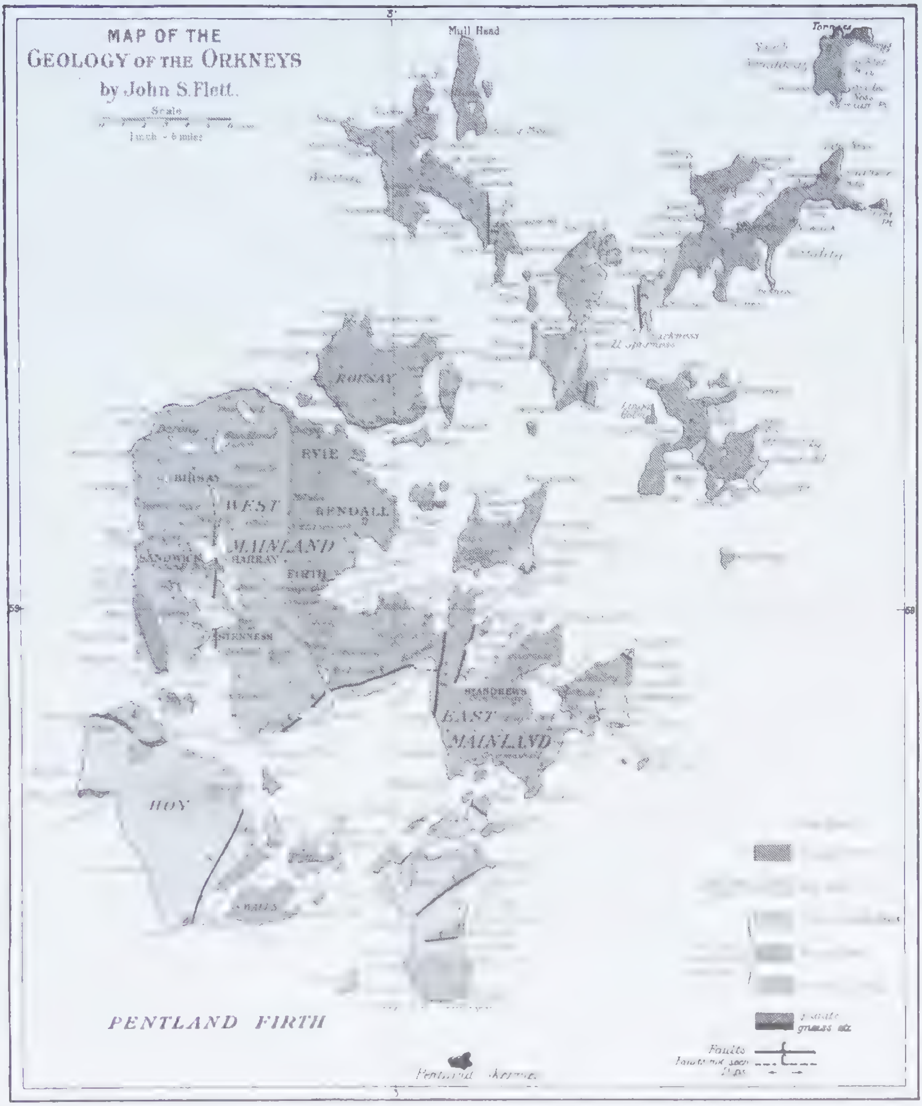
Analysis of Nickel.

Manganese, ...	1·63 per cent.
Magnesium, ...	0·28 ,,
Iron, ...	0·75 ,,
Nickel, ...	97·22 ,,
Total, . . .	99·88

The FORBES's method experiments were carried out in Edinburgh University Physical Laboratory; the other method was done in the Physical Laboratory of the University College of North Wales, and from time to time the work was carried on partly in Edinburgh, partly in Bangor. I have to thank Professor TAIT and Professor GRAY for affording me every facility in carrying out these determinations.

EXPLANATION OF FIGURES.

Fig. 2. A. Gas supply.
B. By-pass.
C. Tube leading gas to burner.
D. Opening for letting out mercury to regulate temperature of cut-off.
E Place at which mercury acts on gas supply.
F. Glass flask containing mercury.
G. Argand gas burner.
H. Mercury.
J. Cast-iron pot.
K. End of bar heated.
L. Thermometers.
M. Bar of Nickel.
N. Brass cap.
O. Water inlet with thermometer.
P. Water outlet with thermometer.
Q. Flask for measuring rate of flow of water.
R. Inverted bottle } on reduced scale.
S. Water tank }
Fig. 3. A, B, C. Tubulures for thermometers containing Wood's alloy.
D. Gas burner.
E. Chamber of water or naphthalene.
K. Open end of apparatus.
L, M. Asbestos screens.


MAP OF THE
GEOLOGY OF THE ORKNEYS
by John S. Flett.
Scale
Mull Head
ROUSAY
EVIE
BIRSAY
WEST
MAINLAND
RENDALL
SANDWICK
HARRAY
FIRTH
STENNESS
EAST
MAINLAND
HOY
WALLS
PENTLAND FIRTH
Faults

XIII.—*The Old Red Sandstone of the Orkneys.* By John S. Flett, M.B., B.Sc. (With a Map.)

(Read 17th January 1898.)

Literature.

The first geologist to examine critically the Old Red Sandstone of the Orkneys seems to have been Professor Jameson, who in 1800 spent six weeks in a mineralogical tour through the county, and so barren did he find the islands, from his point of view, that he counted his journey one of the most uninteresting he had ever made. As yet, the rich store of organic remains which the dark grey flagstones contained had not been brought to light, but the stimulus given to this branch of investigation by the work of Hugh Miller and Agassiz awakened interest in the subject, and we find that a number of collections was formed, especially from the quarries in the neighbourhood of the town of Stromness. Hence when, at a later period (1848), Hugh Miller paid a visit to this district, as narrated in his *Footprints of the Creator, or the Asterolepis of Stromness*, many of the fossils of these rocks were already well known to local collectors, among whom he mentions particularly the late Mr W. Watt of Skaill and Dr Garson of Stromness. Professor Traill of Edinburgh University had for many years been forming a collection, and specimens had been forwarded by him to Agassiz, who makes mention of the fact. Hugh Miller, in the work above cited, and in his *Cruise of the Betsy* (1858), gives a description of his visit to Kirkwall, Stromness, and various parts of the West Mainland, which contains many interesting facts relating to the occurrence and distribution of the fossils in these districts. Further reference to his work will be found in a subsequent part of this paper. The general similarity of the rocks around Stromness to the sandstones of Cromarty and the flagstones of Caithness, as regards the fossils they contained, may be regarded as well established at this date, and the subsequent descriptions of Orcadian specimens contained in Professor M'Coy's *Synopsis of Classification of British Palæozoic Rocks* (1858) served in some measure to confirm this opinion. So far, there had been no attempt to ascertain the structure of the county, but in 1858 Sir R. Murchison* made a brief survey of the islands. He ascertained that there were at least two main types of sedimentary deposits in the Old Red Sandstone of Orkney,—a lower series of flagstones and, overlying them, conformably, as he believed, a series of yellow sandstones, well seen in the island of Hoy. The lower series at Stromness rested, by means of a basement conglomerate, upon an axis of crystalline rock. A great advance was made in 1878 by the appearance of the first part of Sir Archibald Geikie's mono-

* Sir R. Murchison, *Quart. Jour. Geol. Soc.*, vol. xv.

graph on the Old Red Sandstone of Western Europe.* As the result of two visits to Orkney, in which he was accompanied by Mr B. N. PEACH, he pointed out that the yellow sandstones of Hoy did not pass down conformably into the flagstones which form the basis of that island, but were separated from them by a marked unconformity. At the base of the upper sandstones lay a series of contemporaneous lavas and ash beds, which were in all probability erupted from certain 'necks' in the low-lying district at the foot of the Hoy Hills. These rocks he regarded as belonging to the upper Old Red Sandstone. The lower Old Red Sandstone consisted principally of a great thickness of flagstones, with which were interstratified beds of yellow and red sandstone, and occasionally of conglomerate. The fossils belonged exclusively to this lower series; and a table is given, compiled by Mr C. W. PEACH, showing the distribution of fossil fishes in the lower Old Red Sandstone of Lake Orcadie, including those of Orkney so far as known at that time. As Sir ARCHIBALD GEIKIE anticipated, subsequent revision has necessitated "considerable pruning of the fossil lists." The conglomerates around the granite axis of Stromness formed merely a local base, "due to the uprise of an old ridge of rock from the surface of the sheet of water in which these strata were accumulated," and were presumably not on the same horizon as the thick conglomerates on which, in Caithness, the lowest flagstones rest. The sandstones interbedded with the flagstones in South Ronaldshay were regarded as in all probability the northward continuation of the similar rocks at Gill's Bay, Huna, and John o' Groats, on the south side of the Pentland Firth. From a geological point of view, the brief notice of the Old Red Sandstone of the Orkneys contained in this paper forms by far the most important contribution to the knowledge of the subject published up to that time.

In two papers on the Geognosy of Orkney,† published in December 1879, Professor FOSTER HEDDLE showed the existence of a well-marked syncline beginning in the North Isles in the island of Eday, and continuing thence through Shapinshay and Inganess Bay to Scapa and the north-west corner of South Ronaldshay. The beds which occupy the centre of this trough are coarse arenaceous freestones, which rest perfectly conformably on the ordinary blue flags of the islands, and at Heclabir, in Sanday, contain thin beds of conglomerate. These sandstones cannot, in consequence, be the same as the yellow sandstones of Hoy, which unconformably overlie the flags. In these papers many interesting details are given of the minerals occurring in the islands, and of the structural peculiarities of the flagstones, especially as seen in the magnificent coast sections.

In 1880 Messrs PEACH and HORNE made a much more detailed examination of the islands than had previously been attempted, and the result was an important paper on the Old Red Sandstone of Orkney.‡ They showed that in all probability the upper Old Red Sandstone of the district was confined to the island of Hoy, while the rest of the

* Sir ARCH. GEIKIE, "The Old Red Sandstone of Western Europe," pt. i., *Trans. Roy. Soc. Edin.*, vol. xxviii. pp. 409 and 410.

† *Mineralogical Magazine*, "The Geognosy and Mineralogy of Scotland," part v.—Orkney, M. FOSTER HEDDLE, M.D., 1880, p. 102.

‡ *Proc. Roy. Phys. Soc. Edin.* 1880

county consisted of the flagstones and sandstones of the lower series. The distribution of these two members was described, and sections given to show their relation to one another. In their paper on the Glaciation of the Orkney Islands* a map was published, which reappears in the chapter contributed by them to Tudor's *The Orkneys and Shetland*,† and leaves little to be desired so far as regards a knowledge of the distribution of the different lithological types which constitute the Old Red Sandstone of the Orkneys. The structure of the county, they regarded, with Professor HEDDLE, as, in the main, a syncline which runs from Eday to South Ronaldshay, broken in the Mainland by two great faults which cross it and follow the shores of Scapa Bay. In the centre of this syncline lie the sandstones which form the uppermost member of the lower series, while the flagstones form the rest of the district, with the exception of the area occupied by the upper Old Red Sandstone in the island of Hoy. They showed also that in Shapinshay, among the yellow sandstones of the lower Old Red, occurred a belt of contemporaneous volcanic rocks, consisting of a single outflow of a diabasic lava.‡

I.—THE PALÆONTOLOGICAL SUBDIVISIONS OF THE ORCADIAN OLD RED SANDSTONE.

The Eday Sandstones.—So far, those geologists who had endeavoured to make out the structure and succession of the Orcadian Old Red Sandstone had relied mostly on the different types of sedimentary rocks to establish their conclusions, without reference to the fossils the rocks contained. But in 1896, in a paper read to the Royal Physical Society of Edinburgh,§ the present writer showed that among the yellow sandstones of the lower Old Red Sandstone of Deerness, Orkney, occurred three fossils not previously recorded from Orkney, and known only to occur in the John o' Groats sandstones of Caithness, viz., *Dipterus macropterus* (Traq.), *Tristichopterus alatus* (Egert.), and *Microbrachius Dicki* (Traquair). In this way the opinion, already expressed by previous authors,‖ that the sandstones which conformably overlie the flagstones in Orkney were the northern representatives of the similar beds at John o' Groats, Caithness, was confirmed by palæontological evidence. During the following summer investigation was made whether the sandstones in other districts of Orkney, to which had been assigned the same position, contained the same suite of fossils, with the result that in several of the localities examined (in Shapinshay, Inganess Bay, and Eday) one or other of them was proved to occur, and it was established that they constituted the type fossils of a palæontological zone of the Orcadian Old Red Sandstone, which was at the same time distinguished by the lithological characters of its rocks. This may, in consequence, be designated the zone of *Tristichopterus alatus* (Egert.), or, from the locality in Orkney in which they have been principally studied, *the Eday sandstones*.

* *Quart. Jour. Geol. Soc. Lond.*, vol. 36.

† London, 1883.

‡ The occurrence of this basalt was noted by JAMESON, *Mineralogy of the Scottish Isles*, ii. 235.

§ *Proc. Roy. Phys. Soc. Edin.*, vol. xiii.

‖ PEACH and HORNE, *op. cit.* Sir A. GEIKIE *Old Red Sandstone*, p. 409

As will be shown in a subsequent part of this paper, they fall naturally into two subdivisions, a yellow series beneath and a red series above; and it is the thin layers of flag intercalated in the yellow sandstones which have furnished the fossils described. A no less striking characteristic of these beds is the occurrence in them of that zone of volcanic rocks of which the first mention was made by Professor JAMESON.*

The Rousay Beds.—The inquiry was next advanced into the beds which underlie this zone, and were known to consist of a series of flagstones, presumably of great thickness, and of wide distribution throughout the county. All efforts to break up this series into recognisable subdivisions by means of belts of rock, with sufficiently well-developed peculiarities to ensure their recognition in different districts, had hitherto failed; † and, from an extensive knowledge of these rocks, the present writer felt that success was hardly to be hoped for in such an attempt. But should the distribution of their fossils show that certain forms occurred only on particular horizons, this great series could be broken up into zones, which could be identified wherever they occurred, if only they contained a sufficient number of organic remains in a satisfactory state of preservation. The base of the Eday sandstones was chosen as forming a well defined horizon, from which it would be possible to work downwards into the flagstone series in search of type fossils. These underlying beds were then followed from Eday, Westray, and Sanday in the north to South Ronaldshay in the south; the geological structure being carefully mapped, and a record of the fossils observed in each district compiled at the same time. The flagstones of these districts proved to be barren and unfossiliferous compared with the well known localities, chiefly in the West Mainland of Orkney, from which for many years fossils had been obtained in great numbers. Yet in every district decipherable fragments were to be found; and in some localities the fossils were quite as satisfactory as in the better known beds of the West Mainland. By far the most common were the sculptured bones and scales of *Glyptolepis paucidens* (Agassiz), which occurred in every district examined, often in great profusion, and with them *Dipterus valencienesii* (Sedgwick and Múrchison), in every locality, and almost equally abundant. In fact, both these fossils occur right up to the base of the Eday sandstones, though as yet in Orkney not known with certainty to pass up into these overlying rocks. In Deerness, Holm, and Eday the beds immediately below the sandstones are crowded with *Dipterus valencienesii* (Sedgwick and Murchison), often in fine preservation, and covering the surface of whole slabs of rock. After these in frequency comes *Homosteus Milleri* (Traquair), of which the large and usually broken plates are often to be seen. Other fossils were relatively few. In Crook Bay, Shapinshay, I found a Cheiracanthus, which when submitted to Dr TRAQUAIR was determined to be *Cheiracanthus Murchisoni* (Agassiz). At Dingieshowie, Deerness, at Kirkwall, and elsewhere, *Osteolepis macrolepedotus* (Ag.) is found. *Diplopterus Agassizi* (Traill) occurs in the East Mainland, *Estheria membranacea* at Kirkwall, Rendall, and Westray. *Coccosteus decipiens* (Ag.) at Kirkwall, Dingieshowie, and even in the sandstones at Deerness, as I learned from Mr

* *Op. cit.*

† ARCHIBALD GEIKIE, *op. cit.*, p. 410.

MAGNUS SPENCE of Deerness, who forwarded a specimen he found in Newark Bay to Dr TRAQUAIR. To these we must add a new and undescribed species of Asterolepis, of which scattered plates were found by Mr SPENCE of Deerness and myself in Deerness, Holm, and South Ronaldshay. These have been presented by us to the Edinburgh Museum of Science and Art, and Dr TRAQUAIR has kindly consented to draw up and publish a description of them. This interesting fossil is, so far as we know at present, confined to a narrow belt of the flagstones immediately underlying the Eday sandstones, where it occurs with *Dipterus valencienesii* (Sedgwick and Murchison), and *Glyptolepis paucidens* (Ag.); and should further investigations confirm this restricted distribution, it may eventually be taken to mark the existence of a palæontological sub-zone immediately beneath that of *Tristichopterus alatus* (Egert.). That already it should be known from three localities widely separated, and in each case from precisely the same horizon, shows that it can hardly be called a rare fossil in Orkney, and in the future further specimens may be confidently expected to turn up should these beds be submitted to careful and extended investigation. With this exception, this list of fossils contains none which is not of very general distribution throughout the whole thickness of the Orcadian flags.

But when, in the progress of the mapping, a layer of rocks occupying a somewhat lower position was reached, fossils were obtained which were new to Orkney, or among the very rarest of those recorded from it. In the island of Rousay I found along the west side a belt of rocks containing *Coccosteus minor* (Miller), the best specimens being obtained in a quarry of thin slaty flagstones near Sacquoy Head. With it occurred the large enamelled scales of a ganoid fish, of which the fragmentary remains were not sufficient for satisfactory determination. Application was made to the proprietor of the island, General Burroughs, for liberty to quarry, and permission was at once granted. Better material was thus procured, and all doubt removed by the discovery of well preserved remains of *Thursius pholidotus* (Traquair), an addition to the list of the fossil fishes of Orkney. Both occurred on the same bed of rock, and are here recorded from Orkney, one for the first time, the other after a lapse of almost forty years, during which the knowledge of its occurrence seems practically to have disappeared. Curiously enough, when, at a subsequent time, at my request, Dr TRAQUAIR examined for me certain plates of *Coccosteus minor* (Miller) preserved in the British Museum,* which, I presumed, had come from another locality mentioned by HUGH MILLER, he informed me that these specimens, which belonged to the Egerton Collection, were derived from the same locality, but when or by whom they were collected is not known. A very careful search, a year or more previously, among all the local collections of fossil fishes, had failed to bring under my notice any remains of this fish, and none seem to have passed through Dr TRAQUAIR's hands, as he comments on its apparent absence from the north side of the Pentland Firth.†

* A. SMITH WOODWARD, B.M. Cat.—Fossil Fishes, pt. ii, p. 291.
† "Achanarras Revisited," *Proc. Roy. Phys. Soc. Edin.*, xii. 285.

Hugh Miller, in his *Cruise of the Betsy* (1858), p. 358, narrates how, during his stay in Kirkwall, he paid a visit to a quarry a few hundred yards to the east of the town, where he observed numerous specimens of a species of Coccosteus, which he regarded as the same as those he had received from the neighbourhood of Thurso (collected by Robert Dick), and as certainly distinct from, and not merely young forms of, the common *Coccosteus decipiens* (Agassiz). For these he extemporises the name of *Coccosteus minor*. As no specimens of this fossil from Orkney were contained in his collection, and no further material had been obtained from this locality for many years, the accuracy of this observation remained open to some doubt, in spite of his careful identification. Unfortunately, these quarries are now practically worked out and deserted, but I can remember, years ago, seeing in the stones of some old houses in Kirkwall, which had evidently come from this quarry, great numbers of very minute specimens of a Coccosteus. With the rediscovery of this species, however, these doubts in great measure are removed; and as I shall subsequently show, the horizon of these rocks in the vicinity of Kirkwall is identical with that of the beds which in Rousay contain the same fossils. Hence, there is every presumption that this is another locality in Orkney for this species.

In the extreme south end of South Ronaldshay, I found at Banks Geo further examples of the same species, and as here they occur at no great distance from the Eday sandstone series of this island, it would seem that the horizon is a somewhat higher one than that in which it occurs in Rousay and in Kirkwall; but as the island is traversed by a number of faults, no very great reliance can be placed on any estimates of the thickness of the intervening rocks.

Here, then, we have from three localities—one in the north, one in the centre, and one in the south of the county, the extreme stations being over thirty miles apart—the occurrence of a distinct and characteristic fossil in the flagstones. With it occurs another *Thursius pholidotus* (Traquair), which is nowhere known except accompanying it. From the many quarries in the West Mainland, from which for seventy years innumerable specimens have been obtained, not one case is known in which these have been found, and it may safely be presumed that there they do not occur. Their absence, at any rate, cannot be accounted for by imperfect preservation or insufficient search. They may be assumed, in consequence, to constitute the type fossils of a zone of the Orcadian Old Red Sandstone beneath that already defined for the Eday sandstones, and the beds in which they occur I shall designate, from the locality in which the fossils are best preserved, *the Rousay beds*.

List of the fossils contained in the Rousay beds of Orkney:—

Thursius pholidotus (Traq.), Rousay.

Coccosteus minor (Miller), Rousay, Kirkwall, S. Ronaldshay.

Glyptolepis paucidens (Ag.), Kirkwall, Rousay, Eday, Tankerness, Westray, Sanday, Evie, etc.

Dipterus valenciennesii (S. and M.), Kirkwall, Tankerness, Rousay, Eday, Evie, Firth, Westray, Sanday, etc.

Homosteus Milleri (Traq.), Kirkwall, Firth, Rousay, Westray, Sanday, Tankerness.
Cheiracanthus Murchisoni (Ag.), Shapinshay.
Coccosteus decipiens (Ag.), Deerness, Tankerness, Kirkwall, S. Ronaldshay.
Osteolepis macrolepidotus (Ag.), Kirkwall, Deerness.
Diplopterus Agassizi (Traill), Toab.
Estheria membranacea, Kirkwall, Rendall, Westray.
Asterolepis, sp. nov., Holm, Deerness, S. Ronaldshay.

The Stromness Beds.—A careful examination of the list above given will show that not only does it include certain fossils new or rare to Orkney, but that certain others well known to occur there are wanting. It may be said that practically all the fossils in the museums of the world or in private collections which have been furnished by the Orkney flagstones come from a restricted district in the West Mainland, and in the vicinity of the town of Stromness. Here the richness in fossil remains, and their fine preservation, is in striking contrast to the Rousay beds which occupy the remainder of the county. And not only are the fossils more numerous, but species occur which have never been obtained from other districts. Of these, there are two species of Pterychthys—*P. Milleri* (Ag.) and *P. productus* (Ag.)—*Cheirolepis Trailli* (Ag.), *Diplacanthus striatus* (Ag.), and *Gyroptychius angustus* (M'Coy). These, then, in turn constitute the type fossils of still another zone of the Old Red of Orkney, which from the locality of their typical development we will call *the Stromness beds.* With them others occur which are present also in the Rousay beds, viz.—

Coccosteus decipiens (Ag.).
Homosteus Milleri (Traquair).
Dipterus valenciennesii (Sedgw. and Murch.).
Osteolepis macrolepidotus (Ag.).
Diplopterus Agassizi (Traill).
Cheiracanthus Murchisoni (Ag.).

No value attaches to these latter as zone fossils, while there can be no doubt that the former, or some of them at any rate, are entitled to this rank. Much remains to be done before the knowledge of the distribution of the various fossil fishes in the Orcadian Old Red Sandstone can be said to be complete, but, from the Stromness beds at any rate, we have the result of seventy years of the activity of collectors, and the main facts must be regarded as already sufficiently established. That in no case have the type fossils of the Rousay beds been obtained in this locality is perfectly certain, and is a striking fact when we remember that the present writer has obtained these species from two localities in other parts of the county (South Ronaldshay and Rousay) in the course of a short space of time; while in no place have the type fossils of the Stromness beds been obtained along with those of the Rousay beds, or, for that matter, in any locality in which, according to the geological structure of the county, these latter are present; and further, as will be subsequently shown, these results, obtained from a study of the distribution of the fossil fishes of Orkney alone, are in substantial

accordance with the facts already known regarding their distribution in the other districts in which they occur. A mutually exclusive occurrence of this nature can only be regarded as due to the disappearance of one series of forms before the arrival or evolution of the other, and clearly establishes that the successive stages of the deposition of the Old Red Sandstone of the Orkneys were accompanied by changes in the fauna which inhabited the waters in which the rocks were being formed.

II.—The Structure of the Orkneys.

I. *Stromness Beds.*

To the geologist who endeavours to unravel the structure of the Orkneys, a magnificent opportunity is afforded by the excellent and numerous coast sections. So completely is the country cut up by sounds and bays, that at no place can there be any doubt as to the general structure; and even in the larger areas of land, as in the West Mainland, wherever cultivation is to be found, dwelling-houses and stone dykes have been built, and one is, as a rule, at no difficulty in finding stone quarries within a comparatively short distance of one another. If we add to these the many opportunities provided by the inland lochs and streams for an examination of the underlying rocks, it will readily be understood how it is possible, in a comparatively short time, to map with satisfactory detail very considerable areas of country. Only in a very few places do superficial accumulations of boulder clay or peat moss conceal the relations of the rocks beneath, through any extensive tract of land. Wherever the flagstones are present, the structure may almost be said to be writ large on the face of the country. As has been frequently observed by writers on the scenery and geology of Orkney, the hills have then markedly terraced contours, the harder beds of flag resisting erosion and forming a terrace, while the softer beds between, by their more rapid decay, form miniature escarpments. These terraces are everywhere present in flagstone districts of Orkney, and to the experienced eye at once reveal the secret of the underlying structure. In some places, as in Rousay and in Westray, they form so noticeable a feature of the landscape, as to remind one at once of the terraced volcanic districts of many parts, both of Eastern and of Western Scotland. That they are preglacial in origin is proved by the glacial striations with which they are often covered,* and no doubt they have suffered during that epoch a considerable amount of rounding and obliteration; their fine development on the west side of Rousay and of Westray is thus a relic of the old preglacial Orcadian landscapes, which owes its preservation to the fact that the ice movement being from east to west, the west side of these hills was spared the intense erosion to which the rest of the country was being subjected.

The Stromness beds of Orkney, although, as a matter of fact, probably the least extensively developed of any of the subdivisions of the lower Old Red Sandstone, have,

* Peach and Horne, *Proc. Roy. Phys. Soc., Edin.*, 1880, p. 3.

curiously enough, received hitherto by far the greatest share of attention. This is due, without doubt, to the number and excellent preservation of their fossils, of which HUGH MILLER was led to make the somewhat hyperbolical statement, that were the trade once fairly opened, they could supply with ichthyolites, by the ton and by the shipload, all the museums of the world.* The list of collectors who have searched these beds is a long one, and includes many eminent names,—HUGH MILLER, Professor TRAILL, Mr C. W. PEACH, Mr W. WATT of Breckness, the Rev. J. H. POLLEXFEN, Dr CLOUSTON, to mention only a few of those who, in a previous generation, were the first to develop their palæontological resources. The district to which they are confined is compact and of no great area, lying mostly in the West Mainland, in the parishes of Stromness, Sandwick, Birsay, and Harray. If to this we add the flagstones which unconformably underlie the sandstones of the west end of Hoy, and those also around the granite area in Graemsay, we include the entire district from which have been obtained the many Orkney fossils which are deposited in the museums of the world. The rest of Orkney is a district relatively barren and uninteresting to the collector, with the exception of certain areas of the Eday sandstones, such as Deerness—where, indeed, the abundance of the fossils hardly compensates for the paucity of specific forms.

The granite of Stromness.—Professor JAMESON seems to have been the first to recognise the relation between the ancient crystalline rocks of the granite axis of Stromness and the flagstones of Old Red Age which rest on them by means of a thin basal conglomerate. As it has already been more than once described, a brief notice here will suffice. The area occupied is elliptical in shape, and stretches from the Ness of Stromness to the Point of Inganess on the west coast, a distance north-west of about five miles, with a breadth of about a mile. In the hand specimen it is mostly a pink, sometimes a grey granite, of medium grain, and with only a black mica. In many places it is markedly schistose, as at the Ness of Stromness and behind the town, sometimes passing even into a flaggy garnetiferous† mica schist. Numerous veins traverse it, fine-grained elvans and quartz porphyries, with stony matrix and large quartz phenocrysts, and very coarse pegmatites, usually without mica, and showing traces of graphic structure. The microscope shows the rock to be a pretty normal granitite, with orthoclase, plagioclase, and microcline (in small quantities), quartz, biotite, and, especially in the segregation veins, occasional micropegmatite. Sections cut from the gneiss show it to be of similar constitution, but the pressure twinning of the polysynthetic felspars and the strain shadows in the quartz show that in these bands the rock has been subjected to a deforming force.

The basal conglomerates.—Wherever the actual contact between the granite and the flags is exposed, it proves to be an unconformable junction, the rock immediately resting on the granite being always a conglomerate composed of fragments of the crystalline rock. Admirable sections are to be obtained at the Ness of Stromness and

* HUGH MILLER, *Footprints of the Creator*, p. 2.

† HEDDLE, *Geognosy of Scotland*—'Orkney,' p. 135.

at the Point of Inganess. Both have been frequently described, and of the latter locality Professor HEDDLE has given a map. The granite conglomerate is also seen at the Point of Ness, and in the flag quarry at Garson Burn on the Kirkwall road. In no case is it of any considerable thickness, 30 feet being probably the greatest depth anywhere exposed. With it are mixed sandy flags and coarse arkoses, but it is not a little remarkable how soon it gives place to a normal fine-grained dark grey flag, exactly similar to those which cover such wide districts of the county. In fact, such flags are in many places interbedded with layers of a coarse conglomerate. At Yeskenaby, near Inganess, occurs a series of beds of a coarse sandy millstone grit, in which there is a well known quarry for millstones; and though its junction with the granite and conglomerate of Inganess is by means of a small fault, it is easy to see that it is really the rock just overlying the conglomerate let down by this fault against the granite. In fact, on the north-west corner of Inganess, similar beds occur in the cliff where they rest on the granite and granite conglomerate, which form the low shore below. This is in Orkney the only representative of the thick sandstones which elsewhere rest on the basal conglomerate, a fact which strongly supports Sir A. GEIKIE's opinion that the granite axis of Stromness is a mere local base. Yet the shores on which these fine flags were laid down must have been tranquil and tideless, as deposits so fine could not possibly rest on an exposed or tide-swept shore. The innumerable sun-cracked and ripple-marked surfaces everywhere present in the Orkney flags show that they are the accumulations of a shallow sea, yet they can hardly be regarded as littoral deposits; they were rather the finer sediment of landlocked areas of fresh water, in which the coarser material rapidly sank to the bottom, and was deposited immediately around the river mouths.

The Stromness flags.—The flagstones of the Stromness series encircle this granite and conglomerate, and are beautifully exposed in the magnificent sections of the west coast of the Mainland of Orkney, from the Ness of Stromness to the Brough of Birsay. This most interesting coast has been described by almost every writer on the geology of Orkney. Many of the well known localities for Orcadian fossils occur along this shore (*e.g.*, Rocket House, Breckness, Belyacroo, Ramnageo, Quoyloo). Starting from Stromness we find the rocks have a westerly dip along the shore to Breckness, W.S.W., then along the Black Craig, W.S.W., at Yeskenaby, W.N.W., at Skaill, W. and N.W., and, north of Skaill Bay along Outshore Point to Marwick Head and the Brough of Birsay, about N.W. for almost the whole way. The dips roll somewhat, being S.W., W., and N.W., as is best seen between Inganess and Skaill Bay, but everywhere there is a persistent westerly component. Here we are, in fact, on the west side of a great anticline, which forms the chief structural feature of the West Mainland of Orkney. For about four miles back from the cliff, in all the quarries and burns of Stromness, Sandwick, and Birsay, there is the same universal westward dip. The anticlinal axis runs approximately from Waulkmill Bay in the south to Crustan Point, a mile west of the Brough of Birsay, for to the east of this line, in Firth, Harray, and Evie, easterly dips are consistently

present. The long axis of the Loch of Harray corresponds very closely with the crest of the anticline, as on the different sides of the loch the dips are opposite, and at Brodgar Bridge, at Ness in Harray, and at Dounby we have the flat or gently rolling beds which occupy the summit of the arch. A transverse section of the anticline is exposed on the north coast of the Mainland, from Marwick Head in Birsay to Costa Head in Evie. At Marwick Head the dip is N.W. about 10°, and this continues, with occasional variation and a few small faults, seen in the Bay of Birsay, to Skip Geo, just east of the Brough. Thereafter, along the coast by Crustan to the mouth of Swannay Burn, the rocks lie very flat, with gentle and frequently changing dips, in which, on the whole, those to the east and north-east preponderate. In Costa Head the east dip is persistent, and, gentle at first, constantly increases along the shore line to Burgar, and thence to Aikerness Point in Evie. In this entire and perfect section no disturbance of the flags is anywhere seen sufficient to indicate the existence of a fault of any importance.

If we traverse the Mainland along an east and west line through its centre, the result is the same. Starting at Skaill Bay, we find that the rocks are rolling, but the dips are always westward. Between this and Dounby the low lands are in many places covered with boulder clay, but in all the quarries the dips are west till we arrive within a few yards of the village, where it rolls to north-east. More exposures can be examined by following a line past the Loch of Clumly to the Bridge of Brodgar, which separates the Lochs of Harray and Stenness, as along this line there is an abundance of stone quarries, and the loch shore yields valuable natural sections. At Aith, W., at Sandwick Manse, W. 10° N., at Clumly Loch, W., at Lyking, W. 10° N., finally at Bookan, in one of the most prolific in fossils of all the quarries in Orkney, we have an unbroken chain of west dips, which ends only in the isthmus on which are placed the Standing Stones of Stenness. Along the shore of the Harray Loch, from Voy to Brodgar, the section is very complete, and not quite so simple as the inland exposures would have led us to expect. The rocks which form the Ness of Tenston have indeed a prevalent west dip, but sometimes roll to the east, while reefs of vertical beds run out into the loch in a direction N. 10° W., and everywhere there is much contortion and slickensiding, the organic matter of the dark flags having been deposited as a brightly polished layer on the bedding planes. These are the symptoms which everywhere in Orkney indicate the presence of a considerable fault; and as these broken rocks of Tenston Ness occupy a belt of the breadth of about half a mile, the dislocation can hardly be supposed to be a trivial one. Traced southwards, the same phenomena are to be seen in the rocks around the Bridge of Waithe. From Garson farm, near Stromness, by Bu Point, to the Bridge of Waithe, the rocks are folded into many sharp little anticlines and synclines, with mostly a north and south strike. At the bridge and down the Ireland shore by Cumaness, reefs of vertical slickensided and crushed rock are seen in several places running N. 10° W., and from here along the shore to Houton we have again a continual and rapidly changing succession of little folds (as was remarked by Messrs

Peach and Horne*). Along the shores of the Stenness Loch from Onston to below Deepdale, the same phenomena are repeated. Yet in this district the amount of actual crushing and fracture is much less than on Tenston Ness, and there can be no doubt that the throw of the fault is rapidly diminishing as it passes south. Similarly, to the north, on the shore of the Harray Loch at Kirkness, these appearances are repeated, and no doubt the fault runs northward to the west of Dounby village, though here not easily traceable, owing to the thick boulder clay sheet which covers these low grounds. Here, too, it is dying out, and no trace of it is to be found on the north shore of the Mainland.

Continuing our traverse across this fault, we find that the persistent west dips practically cease at the Standing Stones, where, for a time, the beds are gently rolling, and they are last seen in the quarries to the north-west of Maeshowe. In all Harray the dips are gently eastwards, except on the shore of the loch at the Point of Ness, and these east dips continue through the whole of the range of hills which, starting at Finstown, runs northwards to Costa Hill in Evie, and separates the parishes of Birsay and Harray from Evie, Rendall, and Firth. Similarly, in Greenay Hill, Birsay, in Hunland, and in the hills to the east of the village of Dounby, the easterly dips prevail. It is only in the extreme east of the Mainland, in Woodwick, Evie, in Rendall, and in several places along the shores of Firth Bay, that this direction is reversed, the rocks of this district having in many places a very gentle inclination to the west, and forming thus a little marked syncline.

Such being in its main features the structure of the West Mainland of Orkney, we would naturally expect to find the Sandwick and Stromness beds repeated on the eastern limb of the anticline in Harray and Stenness. This, however, is not the case, as the richly fossiliferous beds on the west side of the Stenness lochs do not reappear on the east, where the rocks in many points resemble the Rousay beds of the North Isles and the East Mainland. They are comparatively poor in fossil remains, and have never yielded, to my knowledge, the type fossils of the Stromness zone. This is, there can be little doubt, the effect of the north and south fault, which has let down these comparatively barren beds against the Stromness series which encircles the granite axis. It is only in the northern part of this area, at Dounby, Greenay Hill, and other localities in Birsay, that the fossils of the Stromness beds are to be found in quarries with an easterly dip, and here the evidence points to the theory that the fault is rapidly dying out, as it passes northwards to the west of Dounby. The Firth and Harray beds may be, in consequence, relegated to the passage beds between the Stromness and the Rousay series of the Old Red of the Orkneys, and seem to be on the same horizon as those which occupy the wide area which stretches from Stenness, through Orphir, into Kirkwall. As we shall see later, when we continue the section through Rousay and Egilshay into the Eday sandstones, we have a constantly ascending succession; and as nowhere do the Stromness fossils recur, the inference is obvious—as might have been anticipated from

* Peach and Horne, *Old Red Sandstone of Orkney*, p. 10

the fact that at Stromness they rest upon the granite axis—that *the Stromness beds form the lowest zone* of the Old Red Sandstone of the Orkneys.

It is a matter of great difficulty to form a reliable estimate of the thickness of this series in Orkney, as will be evident when we consider that its true base is nowhere seen, and that its upper boundary must, in our ignorance of any but the general facts regarding the distribution of fossils throughout the county, be of necessity an arbitrary one. By far the best continuous section of these beds is that exposed along the shore from the Ness of Stromness to Breckness, nearly three miles to the westward. The section runs in a W. or W.N.W. direction, and during its whole course there is a continuous exposure of the rocks at low water. They dip along the shore about W. 10° S., and during the first half of the distance the average amount of dip is 15°. In the little sandy bay beyond the churchyard the dip swings southwards, and is more gentle for a little, but on the west side resumes its previous direction and amount. If we draw a line perpendicular to the strike and measure the distance, it is almost exactly two miles, and the thickness, allowing for an average dip of 12°, is about 2000 feet, which is exactly the thickness estimated by Sir A. Geikie for a section parallel to this and a mile further south, from the centre of Graemsay to the base of the Hoy Hills.* As a matter of fact, as the flagstones at Ness rest on the granite conglomerate, and the rocks at Breckness, if prolonged northwards along their strike, are seen to be on a level not greatly differing from those which at Inganess rest on the west end of the same granite axis, we are led to the conclusion that the western conglomerates must be on a much higher level than those at the east end of the granite outcrop. But the lowest rocks in this district must be those which have been uplifted by the Tenston fault along the axis of the West Mainland anticline. This fault is prolonged southwards through the Bay of Ireland; and if we carry the section backwards from Stromness to Bu Point, we find that along this shore the rocks are so rolling that no great thickness is required to be added to our estimate, the same beds being probably again and again repeated by means of gentle folds.

Results in substantial accordance with this are obtained by taking a section some six miles to the northward, from the fault on Tenston Ness on the Loch of Harray, to Skaill Bay on the west shore of the Mainland. The length of a section from Tenston due west to the Atlantic is nearly four miles, and in the intervening country the dips never vary greatly from a true W. In amount they differ, being 12° or more at Lyking, at Voy nearly flat, at Sandwick manse 5°, at Rango 5°, at Skaill 3 to 7°. If we accept 5° as an average, the thickness is 1760 feet. In this case the conditions are not so satisfactory as in the preceding, the exposure of rock not being a continuous one.

To this must now be added the rocks which lie between those of Breckness and Skaill and the base of the Rousay series. That at both these places we are well within the Stromness zone is evident from the fact that they are among the best known localities for its type fossils. The district in the N.E. corner of the West Mainland

* Sir A. Geikie, *Old Red Sandstone*, pt. i. p. 410.

(Birsay and Evie) will, in my opinion, be found the most suitable for this purpose. If we take a section from Crustan Point in Birsay, the centre of the West Mainland anticline, to Burgar in Evie, where we cannot be far from the level of those beds which in the west of Rousay contain *Thursius pholidotus* (Traq.) and *Coccosteus minor* (Miller), and strike southwards across the narrow Eynhallow Sound, the total distance is five miles, measured across the strike of the beds. The dips throughout are eastwards, and their average amount is about 3°. There is no evidence of any important fault. The thickness must in consequence be about 1300 feet. The exact position of the Crustan beds in the Stromness series is difficult to fix, but, as along the western shore from Skaill Bay by Outshore Point to the Brough of Birsay, the dips are mostly N.W., as we travel northwards the section is a constantly ascending one, and the beds which occupy the centre of the anticline at the northern shore must be far higher in the series than those which occupy a similar position in the neighbourhood of the Harray Loch. The Crustan beds in consequence are, in all probability, on a similar level to those in the vicinity of Skaill Bay; and if we add the lower half of the thickness between Crustan and Burgar to that from Tenston to Skaill, we obtain a total thickness of about 2500 feet for the Stromness beds of Orkney. The beds of Evie may, on the other hand, be relegated to the basal part of the Rousay series, and as yet there is no palæontological evidence to prevent such a step. These passage beds, in fact, between the Stromness and Birsay series below, and the Rousay beds above, are comparatively unfossiliferous, and have yielded little of value to the most careful search.

SECT. 1.—From Skaill Head (Sandwick) to Start Point (Sanday).

II. *The Rousay Beds.*

The Rousay beds of Orkney lie mostly to the north and east of the county, where they cover a much more extensive area than the better known Stromness series. As yet, however, little attention has been paid to them and their fossil contents, and the scarcity and imperfect state of their fossils is indeed disappointing to one who has been accustomed to investigate the West Mainland beds. One may travel for days along the shores or among the quarries on this group of rocks without bringing home more than one or two imperfect specimens. Yet they are never entirely barren, and careful search is always rewarded with recognisable organic remains, usually scattered bones and scales, while in a few places we may find even entire fishes, as perfect in every detail as those which abound in certain of the quarries of Sandwick and Stromness. Very characteristic of these rocks are the scattered bones, the teeth, and sculptured scales of

Glyptolepis paucidens (Ag.), and with it *Homosteus Milleri* (Traq.) is the most abundant fossil,—if we except only the head plates and scattered fragments of *Dipterus valencienesii* (Sedgw. and Murch.). But the last is quite as common, and probably commoner, in the Orcadian beds, while the two former have certainly their principal seat in the beds now to be described. With these a not unfrequent fossil is the little crustacean *Estheria membranacea*, which, as at Thurso, sometimes covers the whole surface of slabs of rocks, and is, so far as I know, confined to this zone. Other fishes occur—*Cheiracanthus*, sp., *Osteolepis macrolepidotus* (Ag.), *Diplopterus Agassizi* (Traill), *Coccosteus decipiens* (Ag.); but their principal development seems to have been in a previous time, as they are much more numerous in the lower series. Of the fishes peculiar to this zone, *Coccosteus minor* (Miller) can hardly be said to be rare, seeing that already we know it in three separate and widely distant localities. It is a very suitable fossil for zonal work, as even its scattered bones are so characteristic as to establish its identity readily. Of the different species of *Thursius*, only one is as yet known to occur; and indeed, until a means is discovered for diagnosing these fishes from scattered head plates, bones, or scales, it is unlikely that they will ever be recognised as common fishes in this region of Orkney. The state of preservation requires, in their case, to be much more perfect than holds good as a rule of the fossils of these rocks.

The North Isles District.

If we now continue eastwards our section through Orkney from Evie, through Rousay and Egilshay (sect. 1), we find that in Eynhallow the east dips which prevail in Evie are repeated, and these beds strike evidently across the narrow Eynhallow Sound into the west side of Rousay. In the latter island the east dips which mark this side of the great West Mainland anticline may be said to prevail throughout, but are everywhere very gentle, and are occasionally subjected to a temporary reversal. The terraced faces of the hills, most marked on the west side, show at a glance the simple structure and the almost horizontal disposition of the beds. Along the western coast, the dips are gentle and frequently changing, being mostly north and north-east in the northern half and south and south-west near Westness, but from Hullion along the south coast to Avalshay the dips are very persistently east, except for a brief space below Trumland House, where a very insignificant anticline occurs. East dips are constant on the shore of Rousay Sound. On the north shore the magnificent range of cliffs from Sacquoy Head to the Knee of Rousay around the whole shore of Saviskail Bay exposes an ideal section, which shows a structure slightly more complicated than that seen on the south side of the island. On Sacquoy Head the dips are east, but on Saviskail Head a small anticline, on the south shore of Saviskail Bay another, and in Scockness a third, throw the rocks into gently undulating folds, whose axis is nearly north and south, without anywhere a dislocation of any importance. The island is thus a geological plateau, out of which the agents of denudation have carved the valleys and modelled the surface features. Its heather-clad

hills are the highest in the North Isles of Orkney, rising to heights of over 800 feet; and if we allow 1000 feet for the total thickness of rock exposed, we have an estimate which cannot be far from the truth. Few fossils are yet known from it: *Dipterus valencienesii* (Sedgw. and Murch.), *Homosteus Milleri* (Traq.), *Glyptolepis paucidens* (Ag.), with the characteristic fossils *Thursius pholidotus* (Traq.) and *Coccosteus minor* (Miller). These latter occur in what are about the lowest beds of the island, a belt of thin blue calcareous flags seen best at Sacquoy Head on the north-west corner, and striking southwards through the island, to outcrop again at the Taing of Tratland and the adjoining shore. At Sacquoy Head they overlie a bed of conglomeratic sandstone, with pebbles up to the size of a walnut, of gneiss and quartzite mostly, and resembling thus the rocks of Heclabir, to be subsequently described. In Egilshay the easterly dip continues, but here much steeper, with evident crushing and fracture of the rocks;* and I think it likely that through this island passes a line of dislocation, evidence of which is to be found in the Galt of Shapinshay to the south, and in the district of Rackwick in Westray to the north, in both of which places the appearances point to a similar disturbance. This would, in fact, be a north and south fault, skirting the Eday syncline, like that already described in the West Mainland anticline, and those also described in several places by Peach and Horne (Sanday, Berstane, Holm).

If the section be now continued across the Westray Firth to Eday, we find, as described by Peach and Horne,† a strip of flagstones, with a very steep easterly dip, ranging from Ferstness to Sealskerry, and bounding on the west the area of the Eday sandstones. These lie in the trough already described by these authors; and, as they showed, the only other flagstone area in the island is one which stretches from Warness to the Graand on the south shore, and thence N.N.E. to the Kirk of Skaill and the inner corner of Backaland Bay on the east side. As the centre of the syncline runs from Zoar in Sealskerry to Calf Sound in the north, these flagstones have a W.N.W. dip of about 15°, and they have been brought up by a small fault against the red sandstones which occupy the south-east corner of the island.

In Sanday the yellow and red sandstones occupy the south-east end, as shown by Prof. Heddle,‡ broken by a fault which, running north and south through Spurness Promontory, brings up again for a brief space the underlying dark grey flags.§ Beyond them, to the north and east, the whole island consists of flags which form a well-marked anticline, their westerly members dipping to the west like the Eday beds, under which they pass, but arching over on the south shore of Otterswick Bay and near Geramont House, so that at Taftsness, Newark, and the Start the prevalent dips are to the east. These Sanday flags yielded little of value to my search, *Glyptolepis paucidens* (Ag.), *Dipterus valencienesii* (Sedgw. and Murch.), with a few well preserved fragments of an Osteolepid fish being all I noticed. There can be no doubt that they are a repetition of

* Noted by Jameson, *Scottish Isles*, ii. 239.

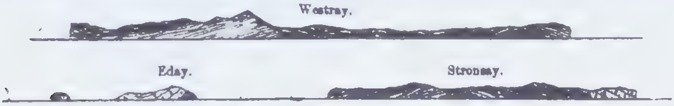
† Peach and Horne, *Old Red Sandstone of Orkney*, pp. 8 and 9.

‡ Heddle, *Geognosy of Scotland*, part v. p. 101.

§ Peach and Horne, *Old Red Sandstone of Orkney*, p. 7.

the Rousay beds on the east side of the Eday syncline, though as yet they have not yielded the characteristic fossils. In the same group must also be placed the North Ronaldshay flags, which time did not permit me to visit and examine in detail. Professor HEDDLE tells us that here the east and west dips are about equally common.* The island of Stronsay, which lies to the south of Sanday, has on the whole a similar structure. It consists for the most part of flags, with one or two areas of John o' Groats sandstones in the south and south-east. The dips along Linga Sound and the north-west side generally are to the north-west, while on the south side, near Housebay, they roll over to the south-east (sect. 2). The structure is thus an anticline like that of the more northern island. I was not able to obtain any data as to the fossils they contain.

In Westray, as PEACH and HORNE † remarked, the structure again is an anticline, though a careful examination showed it was not a simple one (sect. 2). The axis runs from Garth in Tuquoy Bay, to the Sneuk on the north shore. To the west of this, the flags have a persistent though gentle dip to the westward, only reversed for a short space at Noup and Noup Head. To the east of this line the flags form a rolling series, as is well seen along the south shore, where two or three small anticlines and synclines succeed one



SECTION 2.—N.E. and S.W. from Noup Head (Westray) to Lamb Head (Stronsay). 3 miles = 1 inch.

another. On the north shore, the dips are similarly rolling. From the Point of Tafts along the west shore of Rackwick runs a line of dislocation already mentioned as probably a continuation of that seen in Egilshay, and in Rapness the dips are mostly east, though in the extreme south end the flags on the western shore have a west dip. If we neglect the fault, the same strata are thus constantly repeated. There is no doubt they are the same as those of Rousay and of Sanday, the structure being only a continuation northward of that already seen in the northern shore of Rousay. The fossils I found there were *Glyptolepis paucidens* (Ag.), *Homosteus Milleri* (Traq.), *Dipterus valenciennesii* (Sedgw. and Murch.), *Osteolepis*? *Estheria membranacea.*

In Shapinshay we have the two series of rocks—the Rousay beds in the north and west, and an area of Eday sandstones in the south and east. On the east side the beds have a strong south-east dip, but on the north-west corner, around the Galt and in Veantrow Bay, the dips roll greatly, and this is probably the effect of a series of faults which disturbs them : one seems to run from the Galt in the north to the Telegraph hut near Elswick on the south, while the fault which starts at Howquoy Head and runs under the town of Kirkwall must pass just to the west of the shore of the island. As has been pointed out by PEACH and HORNE,‡ the area of sandstones on the south-east

* HEDDLE, *op. cit.*, p. 122. † PEACH and HORNE, *op. cit.*, p. 2. ‡ PEACH and HORNE, *op. cit.*, p. 9.

of the island is probably a continuation southward of the rocks which occupy the centre of the Eday syncline, and the eastward dipping flags of Shapinshay will then correspond to those of Egilshay, Ferstness, and Westray, as the west dipping flags in Stronsay correspond with those of Sanday.

As will be evident from this brief summary, the North Isles of Orkney are composed of two members of the Old Red Sandstone—the Rousay beds and the Eday sandstones. The chief structural feature is the Eday syncline. The Rousay beds of Rousay and Westray, dipping eastwards, pass under the sandstones, and emerge again with a westward dip in Sanday and Stronsay, only to roll over again before they finally disappear beneath the waters of the North Sea. The beds which in Rousay contain the type fossils are, in all probability, the lowest of the flags of this area; and although the sections are frequently interrupted by the sounds which separate the islands, and by not a few important faults, it is quite evident that the entire thickness of rock required to explain the geological facts is by no means great. We have already stated 1000 feet as the maximum required for the Rousay flags, and in no other island is so great a thickness exposed. If we add to these the upper half of our estimate for the east side of the Birsay and Evie series, we have a total thickness along this section of not more than 1500 feet. No more than an approximate estimate can possibly be formed in this district, as the sections are so broken up by water, and no recognisable subdivisions can be established, either lithologically or palæontologically, with which we might ascertain the throw of the respective faults.

SECTION 8.—From Stromness, through Kirkwall, to Roseness (Holm).

The West Mainland District.

Owing to the prevalent north and south strike, the rocks of Rousay may be expected to cross into Evie and Rendall, where they lie in very nearly horizontal but slightly rolling folds, and from here to pass southwards into the district between Finstown and Kirkwall. A similar conclusion is arrived at by the examination of the rocks which stretch eastward from the Bridge of Waithe in Stenness (sect. 3). Here we are among the rolling beds which mark the termination to the south of the fault which runs along the side of the western anticline. These beds are undoubtedly to be placed, with their more northern representatives in Harray, among the upper beds of the Stromness series. Further east in Stenness we find the effects of the western anticline, though here little marked, and evidently dying out. Through most of Stenness and throughout the Ward Hill of Orphir the dips are south-west. The anti-

clinal axis passes almost through Maeshowe down Summersdale into the Kirbuster district of Orphir. In the Heddle Hills of Firth, to the east of this line, the dips are mostly east and north-east, very gentle in the flag quarries, now disused, which crown the hills on both sides of Finstown. From the latter village to Kirkwall we have a rolling succession of gentle anticlines and synclines with axes striking north and south; seen well in the shores of Firth and Kirkwall Bays, where the same beds crop out again and again. There are no steep dips and no traces of any important dislocation, but from Summersdale to Kirkwall, on the whole, the dip is eastward, and we are ascending very gradually in the series. In the quarries to the west of Kirkwall there is a very slight north-west dip, while along the shore to the east of Kirkwall Bay the dips are strongly east. The change is marked by a line of crushed rock which runs under Kirkwall in a N.N.E. direction, and emerges on the shore at Cromwell's Fort. This seems to be the northward continuation of the fault described by PEACH and HORNE as running from Howquoy Head in Holm, northwards along the shore, and forming the eastern boundary of the sandstones of Scapa Flow.* This may be possibly a continuation of that already described as passing through Egilshay into Westray. At any rate it is an important feature in the structure of this part of the Mainland of Orkney, for to the west of it lie the gently rolling beds described, while to the east the dips are steep as a rule, and the rocks thrown into very pronounced folds. In other words, it forms a natural geological boundary to the East Mainland of Orkney.

The East Mainland District.

The second area in which it has been proved that the Rousay group of fossils occurs in Orkney is that around the town of Kirkwall, in which HUGH MILLER remarked their presence more than forty years ago. The structure of the East Mainland has not that simplicity which characterises the West Mainland. To the south-west it is bounded by the fault described by PEACH and HORNE, which brings down the sandstones of Scapa against the flags. The flags along this fault are probably the lowest rocks exposed, for through the whole area there is a constant tendency to a northerly dip, varied, of course, by the subsidiary folds, and the highest rocks occur only in the northern half of the district. Two series of rocks occur—the Eday sandstones in two areas, Berstane Bay and Deerness, the Rousay beds elsewhere.

The structure is clearly defined, an anticlinal axis occupied by the flags passing up the centre of the district in a north-east direction, and forming the Ness of Tankerness, while on each side a syncline brings in the overlying rocks, the sandstones (sect. 3). The section along the public road from Kirkwall to Dingieshowie, Deerness, affords a very good index to the general structure. For a mile or more after we leave Kirkwall, the rocks are steeply inclined to the east and north-east, disturbed, no doubt, by the great fault whose

* PEACH and HORNE, *op. cit.*, p. 11.

outcrop we are crossing, and through the promontory between Kirkwall and Inganess Bay the general dip is to the north-east. At the south-west corner of the latter bay the fault already described by PEACH and HORNE, forming the western boundary of this area of John o' Groats beds, is well seen in the shore, letting down the red sandstones sharply against the blue grey flags. These are the flags which in the old quarries at the East Hill, Kirkwall, rather over a mile away, contain the Thurso fossils, according to the observations of HUGH MILLER.* They form a triangular area between two considerable faults; and though in the land north-east dips prevail, as also along the east shore of Kirkwall Bay, along the northern coast from Carness to Meil Bay, a succession of folds repeats them.

Continuing our section eastwards, we find that the sandstones of Inganess dip north-west to the fault, and at their eastern edges are bounded by grey flags with a similar dip. About five miles from Kirkwall, at Quoyburray, in a quarry near the road, the beds lie nearly quite horizontal, and from that point onwards the dips are south-east and generally steep. The axis of the anticline runs approximately from Sebay Mill to the Ness of Tankerness in an E.N.E. direction, as along this north-west shore of Deersound the dips are slight and rolling; and while, to the east of this, at Yinistay Head and through Tankerness we have the north-west dips, in Deerness these have rolled over to the south-east. At Dingieshowic the yellow sandstones are let down by a fault, but maintain the general south-east dip; and from here, along the shore to the Castle, they lie in a little trough, the dips swinging first to east, then to north-east, when they are succeeded by grey flags, which up to Roseness Point have a north dip. Along the shore of Holm Sound the north and north-east dips show that here, too, we are on the south side of a syncline which runs approximately north-east and south-west, but as we pass westwards beyond Graemshall the rocks are much disturbed, and the dips are inconstant and frequently changing.

In spite, then, of their generally steep dips, the flagstones of the East Mainland are so repeated by these folds that they cannot be regarded as of very considerable thickness, and the disturbance to which they have been subjected renders any estimate exceedingly conjectural. Their fossils are few, yet I found in different places *Glyptolepis paucidens* (Ag.), *Dipterus valenciennesii* (Sedgw. and Murch.), *Osteolepis macrolepidotus* (Ag.), *Coccosteus decipiens* (Ag.), and *Diplopterus Agassizi* (Traill).

It is interesting to observe how the section drawn east and west from the Bridge of Waithe to Roseness, through Kirkwall, repeats the main features of that drawn from Skaill Bay to the Start Point of Sanday (sects. 1 and 3). The Tenston fault passes south through Waithe, and the West Mainland anticline is distinctly to be traced in Summersdale, the rolling beds between Finstown and Kirkwall are those of Rendall and Firth, the Rousay beds recur at Kirkwall, and the broken dislocated flagstones to the east of Kirkwall repeat the structure of the west of Shapinshay and Egilshay. The Eday syncline passes south through Shapinshay to Inganess Bay. The anticline of Tanker-

* *Cruise of the Betsy*, p. 394.

ness is that of Sanday and Stronsay, while the sandstones of Deerness and Holm belong to a syncline unrepresented in the northern section, except it be by the limited areas of yellow and red sandstones in the island of Stronsay.

The South Isles District.

South of the Scapa faults not one of these features reappears, and the South Isles of Orkney form a distinct district, with a well-developed structure of its own. It consists of a geological basin, in the centre of which lie the higher beds, the sandstones.* They form the shores of Scapa Flow, from the Old Kirk of Orphir to near Howquoy Head. They reappear in Hunda, the west of Burray, and the north-west of S. Ronaldshay, here dipping west and north-west, and constitute also the north end of Flotta. Around them pass the underlying flags of Orphir, Holm, the east of Burray, the south-east of S. Ronaldshay, Swona, and the south of Flotta. In the north, the junction is a fault; and through South Ronaldshay and Burray it is evident that several faults run north-east and south-west parallel to the strike of the rocks. Yet in some places the succession is an interrupted one, as, for example, to the west of Grimness Head and in the island of Flotta. In Burray the flags dip west, in S. Ronaldshay north-west, in Flotta north, the strike thus sweeping gradually round. Much broken up as the district is by the sea, it is yet sufficiently clear what the general structure of the whole area must be. The Eday syncline is rapidly dying out in Inganess Bay, and I could find no proof that the yellow sandstones pass across the East Mainland near Kirkwall, to unite with those of Scapa Flow. Even should they ultimately prove to be continuous, it is clear that the broad basin of the South Isles cannot fairly be regarded as a continuation of the Eday syncline, which already at the south end of Inganess Bay has narrowed to less than a mile in breadth, and has, furthermore, to cross the powerful dislocation of the east side of Scapa Bay. In all the features of its structure, the South Isles area shows no point of comparison with that around Kirkwall, still less with that of the North Isles of Orkney.

The largest continuous area of these rocks is that of South Ronaldshay, which alone I had time to examine in detail. It consists of two series, the grey flags of the south-eastern district, and the yellow and red sandstones of the north-west. The general dip throughout is N. to N.W., but the structure is by no means simple, as it is evident from the coast sections that powerful dislocations cross the island from N.E. to S.W. On the west side the flags extend from Brough Head to Barswick, much disturbed in many places; and from thence to St Margaret's Hope, and for a mile further east along Water Sound, the shore consists of yellow and red sandstones (faulted apparently in two places at Barswick, where they are brought down against the flags, and at Sandwick). The Hoxa promontory consists of an anticline of blue flags, and is bounded by a fault which runs across the narrow isthmus. On the east shore, again (sect. 4), the dip is continu-

* Peach and Horne, *op. cit.*, p. 12

ously north, the flagstones stretching from the Old Head to Halcrow Head, whence a small area of sandstones extends to Windwick. Here a fault brings up the flags with a steep north dip, and at Stews Head these again are overlaid by yellow sandstones which stretch along the shore to St Peter's Church, where again the blue flags are faulted up to form the promontory of Grimness and the north-eastern corner of the isle, and to pass conformably into the yellow sandstones along the shores of Water Sound.

Of these rocks the lowest are evidently the flags of Brough Head and Old Head in the southern shore, and here, at Banks Geo, with remains of *Coccosteus decipiens* (Ag.) and of an undetermined osteolepid, I found numerous plates of *Coccosteus minor* (Miller), which have been determined by Dr Traquair. The chief importance of this lies in the fact that it establishes the zonal identity of the flags which encircle the sandstones of Scapa Flow with those which accompany the Eday beds of the North Isles. Here, however, the horizon is, to all appearance, a higher one, as the distance between the *Coccosteus minor* beds and the sandstones of Halcrow Head is not much over a mile; and though there is evidence of faulting in the intervening section, it would seem, as stated by Peach and Horne,* that these faults are not of any great magnitude.

A further interest is lent to the rocks of South Ronaldshay by the occurrence in them of the new species of Asterolepis previously mentioned. Of this I found a plate in a flag quarry on Hest Head. The horizon is that which is, so far as at present known, characteristic of this fish, being in the grey flags about forty feet beneath the base of the Eday sandstones. Another plate of this species was found by Mr Spence of Deerness at the Castle of Claisdic, near Stembuster, in St Andrews, and still another, a year before, by him and myself, a short distance north of Sandside in Deerness. In both these places the geological position is precisely the same; and it seems, in consequence, to be a fish of very restricted vertical range, and may ultimately prove to be the type fossil of a subzone of the Old Red Sandstone of the Orkneys at the top of the Rousay series. That it is to be united with these rather than with the overlying beds is shown by the accompanying fossils, of which the commonest by far is *Dipterus valencienesii* (Sedgw. and Murch.), which occurs often in very great numbers in this particular belt of rock. Remains of osteolepid fishes also occur, but there is no trace of the distinctive fauna of the Eday sandstones.

Lithology of the Flagstones.

When we pass from an examination of their fossil contents to the study of the rocks themselves, at first glance we are apt to be greatly impressed by their monotony, and the endless repetition of beds in no way differing greatly from one another. The effect on Professor Jameson we have already mentioned: his six weeks' journey in Orkney proved the most uninteresting he had ever made. The geologist who is bent on the search for easily recognisable lithological zones which can assist him in the completion of his map is sure to suffer a like disappointment. Immense as is the variety in these beds, no

* Peach and Horne, *op. cit.*, p. 11.

two being in every respect similar, there are yet no recognisable and definite alternations which could with certainty be used in dividing up the whole into an established succession. This is true of the Orkney flags as a whole, as was pointed out by Messrs PEACH and HORNE. They vary greatly, the principal types being a sandy flag, a clay flag or mudstone, and a brittle calcareous or even bituminous flag. The sandy flags never amount to pure sandstones, there being always a certain amount of clay and of silky weathered and bleached mica, with very usually a calcareous cement between the grains of sand. The clay flag is the purest and most abundant type. They are relatively soft, fine-grained, and light grey in colour, except when darkened by organic material. On their bedding planes the pale lustrous mica is often to be seen as a shimmering film, while the microscope shows that in worn, tattered, and crumpled flakes it is an important constituent of their mass. Sand in fine rounded grains and calcite in greater or less abundance are constant constituents. Where these softer beds occur mixed with harder beds on a cliff face, they weather out rapidly into pale grey hollows, and this is the origin of a frequently remarked feature of the Orcadian cliff scenery. The calcareous and bituminous flags are the chief receptacles of the fossil remains inclosed in these rocks. The fossil collector very soon learns that the best specimens are obtained in a brittle, hard, usually slaty and thin-bedded rock, which rings to the hammer like a piece of metal. This is in some measure due to the compactness and impermeability which is conferred on these rocks by their abundant calcareous matter. But there can be no doubt that, in turn, the presence of the organic remains facilitates in some way the accumulation of carbonate of lime in the rock, as frequently around the fossil is a well marked nodule, compact and hard, and evidently calcareous in nature from the rapidity with which it weathers out, leaving the surrounding rock comparatively unaffected. These are especially common in the dark flags among the sandstones of the Eday series. The prevalent colour of these calcareous flags is dark blue-grey, and they are fine-grained, and mostly free from the concretions so abundant in the more argillaceous rocks. In these latter they are so common that hardly a stone could be found without some trace of them. Of all sizes, from that of a melon to less than a pea, and of a remarkable and often grotesque variety of shapes, they show most clearly in the weathered face of an old dry-stone dyke, or on the bare surface at the edge of the high cliffs of the coast. From the manner in which they resist the weather, they are in most cases probably siliceous—they are certainly harder and more difficult to break than the rock surrounding them. Of these concretions the best known example is the horse-tooth rock of Yeskenaby, to which Professor HEDDLE* and other authors have devoted some attention. The rock itself occurs *in situ* at Borwick, near the great trap dyke there. But this is merely an interesting variety of a phenomenon of universal distribution throughout these flags. Their surfaces are often mottled and pitted with innumerable little concretions, which it would be easy to mistake for coprolites or for rain pittings. Not uncommonly these consist of pyrites and of marcasite, which on

* HEDDLE, *op. cit.*, pl. xiv.

weathering give a rusty colour to the surrounding rock. When the flagstones weather, the siliceous concretions, owing to their greater durability, stand out in high relief upon the bedding planes, and give the rock often a curiously fretted and ornamented appearance, and so numerous are they that frequently they resemble a solid mass of fretwork or of repousée ornament upon the surface of the stone. On weathering the flags lose also their prevalent pale or dark grey colours. Many of the dark calcareous flags around Stromness weather with a creamy yellow crust, which resembles that of certain impure carboniferous limestones. Yellow and different shades of brown are the prevalent tints of the weathered stone. The changes are principally the removal of the lime in solution and the oxidation and hydration of the iron. It is the latter which stains the rock, as is seen when we consider the source of the white colour which marks the weathered flags in a peat bed, and which is due to the organic acids of the peat having removed the iron from the rock. The decomposition gradually proceeding inward from the surfaces and cracks, produces sometimes a curious effect on a seashore where a bed of calcareous flag is divided up by many joints into polygonal areas, around the outside of which is a soft, rusty, decomposed film, an inch or more in depth, while the centre area is hard, grey, and comparatively fresh. The innumerable sun-cracked and rippled surfaces were well described by Sir A. GEIKIE * in the flags around Thurso.

In thickness the beds vary from an inch up to perhaps 18 inches. In every district of Orkney, flags of 2 or 3 inches thick and in large flags can be obtained for paving purposes. A favourite kind at present is a coarse sandy flag in thick beds (6 inches), obtained from Orphir. Thinner slabs, used formerly for roofing slates, are also of very wide distribution. The thick beds are valued for building purposes, especially if the bedding planes are smooth and the joints well marked. In the latter case they need no dressing, as the builder places the smooth joint face, often covered with a fine layer of glancing calcite, to the outside of the wall. In some places a variety of flag occurs, very dark in colour and seemingly much crumpled, the minute laminæ of which it consists being contorted in every conceivable fashion. Such beds are of restricted distribution, and usually markedly lenticular, as they thin out abruptly in no great distance. They bear a superficial resemblance to certain curly oil shales in the Edinburgh district, but when broken open they consist of an ordinary grey flag, the contorted layers being often covered with a dark film. They are not due to earth movement and crushing, as they occur in perfectly undisturbed rocks, and they probably result from peculiar conditions of deposit, perhaps the escape of gases or the decomposition of organic matter having produced their irregular internal structure. Where the flags are crossed by a fault the disturbance is often very great, and quite out of proportion to the magnitude of the dislocation. The rocks are bent and twisted, their surfaces slickensided and blackened, or a dark breccia produced, in which the flagstone particles glance with organic matter till they resemble broken bits of coal. In some cases the fault is marked by a layer of crushed rock powder, intensely black in colour, and mixed with calcite and iron pyrites.

* Sir A. GEIKIE, "Old Red Sandstone," *Trans. Roy. Soc. Edin.*, vol. xxviii. p. 393.

The peculiar nature of this flagstone deposit is so strikingly new to the geologist accustomed to the study of other districts that it cannot fail to suggest a consideration of the question of its origin. Sir ARCHIBALD GEIKIE * has insisted strongly on the marked difference between these and the sandstones which in other parts of Scotland are so characteristic of the Old Red. This striking contrast in the nature of the strata points to markedly dissimilar conditions of deposit. As we trace upwards the Old Red Sandstone of the Orkneys, we shall see that in process of time this type of sediment was replaced by the more familiar one of yellow and red sandstones and red marls. There can be no doubt that this was the result of marked changes in the physical geography of the region; and when we remember that at Cromarty beds of yellow sandstone contain precisely the fossils of the flagstone beds around Stromness, and, beyond reasonable doubt, were being formed at the same time, we see clearly the truth of Sir A. GEIKIE'S conclusion that the flagstones of Orkney are merely the result of certain peculiar conditions of deposit. From their rippled and sun-cracked surfaces, they were certainly originally laid down in shallow water; and from the extensive area they now occupy, they must in many cases have been laid down far from land. That this area was tranquil I have shown to be probable, from the way the fine flags lie among the conglomerates of Stromness right against the old granitic shore. A similar mixture of deposits is to be found at the present day only in the land-locked areas of our river mouths and inland lochs. The other striking feature of these flags is the way in which they combine materials in other formations confined to different rocks. All contain sand, clay, and carbonate of lime in varied proportions, yet sandstones, limestones, or true shales are never typically developed in this peculiar formation.

III. *The Eday Sandstones, or John o' Groats Series.*

The Rousay beds of Orkney, as described by many previous writers, pass upwards conformably into an overlying series of yellow and red sandstones and marls, which contain in many places the fossils which characterise the John o' Groats beds of Caithness, and are to be regarded as on the same horizon with them. This is a very different series, and much more varied than the Rousay beds of Orkney. An entire change in the nature of the sedimentary deposits indicates a complete and comparatively rapid change in the physical conditions of the area. The yellow sandstones, with their flag beds grading upwards into red sandstones and marls, must have been the formation of shallow areas of water, with currents sufficiently strong to introduce now and then even layers of coarse gravel. The unvarying and monotonous Rousay beds, the deposit of still, though comparatively shallow water, come suddenly to an end. It is interesting to observe that these changes were accompanied by the outburst of volcanic action in a district which had for ages been the seat of uninterrupted quiet sedimentation. In the whole thickness of the Stromness and Rousay beds of Orkney there is no trace of

* Sir A. GEIKIE, "Old Red Sandstone," p. 363.

contemporaneous volcanic activity. The same conditions prevailed in the Thurso area, as was shown by Sir A. GEIKIE, the first trace of volcanic rocks being the necks on the shore at Huna, which pierce the red beds of the John o' Groats sandstones.* These physical changes heralded also the appearance of a completely new fauna in the district. It is long since it was shown by the late C. W. PEACH that at John o' Groats occurred certain fossils nowhere else to be found, viz., *Tristichopterus alatus* (Egert.) and *Microbracheus Dicki* (Traq.),† and to these *Dipterus macropterus* (Traq.) was subsequently added‡ by Dr TRAQUAIR. The same species occur in Orkney, as I have elsewhere shown, and here they form practically the only known fossils of these beds. With the single exception of a specimen of *Coccosteus decipiens* (Ag.) collected in Newark Bay, Deerness, by Mr MAGNUS SPENCE, and forwarded by him to Dr TRAQUAIR, I know of no other fossils which have been found in them. How sudden and complete the change must have been is shown by the following facts. In Eday, *Glyptolepis paucidens* (Ag.) and *Dipterus valencienesii* (Sedgw. and Murch.) occur within a few feet of the base of the yellow sandstones. In the Deerness district *Asterolepis*, sp. nov., *Osteolepis macrolepidotus* (Ag.), *Dipterus valencienesii* (Sedgw. and Murch.), *Glyptolepis paucidens* (Ag.), and *Coccosteus decipiens* (Ag.) occur in the rocks immediately underlying these beds, *Dipterus valencienesii* (Sedgw. and Murch.) in some places in vast numbers and curiously small in size. With the single exception already mentioned, not one recurs in the richly fossiliferous flags among the yellow sandstones. It would seem as if these species had been unsuited to the new environment in some manner or other, and their extinction had been rapid and complete. The flags so crowded with remains of *Dipterus valencienesii*, only a few of which have attained their full size, irresistibly impress on the mind the idea of a sudden extermination. At a higher level we find the same confused aggregation of fishes in the flagstone belts among the yellow sandstones, but this is on the horizon of the volcanic rocks, and we shall probably be right in regarding it as a consequence of the volcanic activity. The rocks of this series, unlike those they overlie, fall perfectly naturally into two main subdivisions, a yellow below and a red above, the latter possibly an index to the change which ensued on a contraction of the area of the old lake, and rendered it the seat of chemical operations resulting in a new type of deposit.

In their paper on the Old Red Sandstone of Orkney, Messrs PEACH and HORNE described with great accuracy the boundaries of these rocks, which they named the 'upper sandstone series' of the lower Old Red. It will be sufficient if I here give merely a brief account of their distribution. They occur in the centre of the Eday syncline, forming most of the island of Eday and the Red Holm between it and Westray, and lying in a gentle syncline, which is broken by a fault bringing up a strip of flags which stretches from Warness to the Kirk of Skaill. As described by these authors, the

* Sir A. GEIKIE, *Old Red Sandstone*, pt. i. p. 405.

† British Association Meeting at Aberdeen, 1858.

‡ EGERTON, Geological Survey Decade. TRAQUAIR, *Geological Magazine*, Nov. 1888. *Proc. Roy. Phys. Soc. Edin.*, 1896.

succession between the lower and the upper series is a perfectly conformable one. An extension of this syncline occupies Spurness, the S.W. corner of Sanday, and the Calf of Eday. It stretches southwards into Shapinshay, where it forms the south-east corner of the island. These beds have mostly a south-east dip, and belong to the west edge of the syncline. Thence it extends into the opposite shores of the Mainland, and occupies an area which stretches from Holland Head around the shores of Inganess Bay and in a narrow strip to the Skerry of Yinistay in Tankerness. The west boundary of this is a considerable fault already described as seen in the south-west corner of the bay, on the old Kirkwall road, and running thence along the shore and by Berstane House to the centre of the Bay of Meil. On the eastern boundary the sandstones pass perfectly conformably downwards into the flags.

The second area of these rocks is that of Deerness, first described by the present writer in a previous paper. It is separated by the Tankerness anticline from the Inganess Bay area, and the Rousay flags appear on the west corner of Deerness, near Mirkady, and pass up conformably into the John o' Groats beds. The whole area forms a well marked syncline, which includes almost the whole of Deerness, and stretches thenee into Holm, where a narrow area of these rocks surround the farm of Stembuster. The dips throughout the south-east half of the sandstones of Deerness are south and south-east. At Stembuster the south-east dips gradually swing round to E.N.E., and finally to nearly north, near the Castle of Claisdie. Several faults occur in the area, one at the Mull head letting down the red and yellow sandstones against the grey flags, which at Sandside contain *Asterolepis*, sp. nov., and *Dipterus valencienesii* (Sedgw. and Murch.), but none of the John o' Groats fossils. These flags in turn, as we pass southwards, graduate upwards into the yellow sandstones. Another fault must run into Newark Bay (though not seen, the area being occupied by blown sand), for to the east of it the dips are south, while to the west the dips are mostly E.S.E., and the yellow sandstones of one side strike at the red beds on the other. Much of this syncline must lie out to sea, and possibly, as already suggested, the red rocks of Stronsay are really part of it, though it is worth mentioning that the rocks of Copinshay are grey flags, undoubtedly belonging to a lower horizon.

In the south isles of Orkney the sandstones occupy the centre of the basin.* A narrow strip of sandstones bound Scapa Bay from Orphir Kirk to near Scapa Distillery and thence along the eastern shore to Howquoy Head, in Holm. They form the west end of Rousay and the island of Hunda, here dipping west, the north-west corner of South Ronaldshay, with a general north-west dip; and on the east side, at Windwick and St Peter's Church, small areas of sandstones are faulted down among the flags of the south and east side of the island. In Flotta they occupy principally the northern half of the island and the adjacent Calf of Flotta, having here a north dip, and passing down conformably into the grey flags of the southern shore.† Lastly, in the island of Hoy they are found in that part of Walls to the north of Longhope, around the Burn of Ore,

* Peach and Horne, *op. cit.*, pp. 11 and 12.

† Peach and Horne, *op. cit.*, p. 12.

and are separated by a fault by the upper Old Red Sandstone, which extends over the most of the remainder of the island.

The Yellow Sandstones and Flags of the John o' Groats Series.

Starting at the northern extremity of their area in Orkney, we find that in Eday these beds occupy a comparatively small area and are of very limited development. At the Kirk of Skaill, on the eastern shore of Eday, a belt of yellow sandstones immediately overlies the top flags of the Rousay series. These are followed by a thin zone of red marls, which in turn are overlaid by thin-bedded calcareous flags, rich in fossil remains, of which *Dipterus macropterus* was the only one I found in satisfactory preservation. Above these we find a series of yellow and red beds (with thin layers of conglomerate), which form a gradual transition to the red and brown sandstones and marls so largely developed in the centre and north end of the island. The whole thickness of this series is not over 100 feet, and it is, in fact, their most insignificant development in any part of Orkney. Were it not for the very convincing sections elsewhere obtained, it would be impossible to regard these beds as other than a merely local facies of the basal series of the red beds. Messrs PEACH and HORNE* give the following estimated thickness :—

Red and yellow sandstones—
Flagstones, 40 feet.
Reddest shales, 15 feet.
Hard white sandstone, 20 feet.
Gray calcareous flagstones.

—the last being the underlying Rousay series, as I regard them, as they contain no trace of John o' Groats fossils of the group of flagstones interbedded with the sandstones, while *Dipterus valencienesii* and *Glyptolepis paucidens* are not infrequent in them. These yellow beds and flags stretch across London Bay, and emerge again at Millbounds, where the section is very similar to that described.

On the west side of the syncline the same beds crop out again just to the east of Fersness, where they furnish the chief supply of yellow freestone used for building purposes in Kirkwall and throughout the islands. A hundred yards to the east of the pier the yellow beds come in gradually below the red, which here dip about E.S.E. Among them occur again a belt of thin flags and an insignificant red series. The section, in fact, repeats in every respect that to the east, and *D. macropterus* is found in the flags to the west of the pier, but here the thickness must be somewhat greater, as the average dip is about 20°, and the area of shore occupied is about 400 yards. At Warness, again, to the south-west corner of the island, the underlying flags, with here and there a yellow bed, pass up into a yellow sandstone series, 70 to 80 feet thick, over-

* PEACH and HORNE, *op. cit.*, p. 5.

laid by a few feet of red beds, and these by 20 feet of coarse flags (in which I found no fossils). Over these flags, which no doubt are the same as those of London Bay, come a few yellow beds, which rapidly give place to the red sandstones of Sealskerry Bay.

In the south end of Sanday these beds recur, and form the western edge of the promontory of Spurness, disturbed and set on end by a north and south fault, which brings up with them the underlying beds of flagstones in a narrow strip. A thick conglomerate occurs among them at Heclabir, but in other respects they differ little from the Eday sandstones, though, from their limited distribution, no very complete idea of their features can be formed. After we cross the fault above mentioned, we find the red sandstones in great strength, forming the shore to near the Noust of Boloquoy on the north coast. Here yellow and red beds, mixed, strike along the shore, and, slightly faulted at Grunnavi Head, continue with a dip W.N.W. to Blue Geo, where the flags again come in. The thickness here is not great; but owing to the presence of several small faults, an exact estimate is not possible. These beds, tracod along the strike, emerge at Quoyness on the south shore, where, however, they are covered by the blown sand of the beach. The yellow sandstones of Sanday show the same features as those of Eday, and, like them, are of comparatively small thickness.

The conglomerates which occur in these rocks of Eday and Sanday have already been the subject of discussion by several writers.* Professor HEDDLE noted that at Heclabir, in Sanday, occurred a bed of conglomerate about 14 feet in thickness, and that the pebbles it contained consisted of "granites, more than one variety, gneisses, often chloritic, porphyrys, and seemingly of quartzite,—rocks which are entirely different from the primitive rocks near Stromness, and therefore rocks not occurring in the islands." † He states also that both the pebbles and the cementing paste have a highly vitrified aspect, and that he had a strong impression this was a volcanic conglomerate. Messrs PEACH and HORNE state with regard to the beds of Eday, which form very insignificant belts at the base of the red series—nowhere over a few inches in thickness—that "the included pebbles consist of fragments of mica schist, quartzite, gneiss, granite, and other metamorphic rocks, all stained of a reddish colour." ‡ According to my own observations, all those mentioned occur with one exception; the commonest by far at Heclabir being a creamy or white lustrous quartzite, in much rounded and waterworn pebbles, up to 6 inches in diameter. At the latter locality I was unable to find any volcanic rocks, but there were very numerous pebbles of grey limestone, which microscopic sections showed to be entirely holo-crystalline and true marbles, without any trace of organic structure. With these were others which at first puzzled me; but on referring to Mr PEACH, he at once recognised them as cherts and cherty limestones from the Eillean Dhu series of Durness (Cambrian); and the microscope showed that, like these, they were of oolitic structure, though, so far as my examination went, by no means so perfect as in the

* JAMESON, *Mineralogy of the Scottish Isles*, vol. ii. p. 257.
† HEDDLE, *Geognosy of Scotland*, v. p. 103.
‡ PEACH and HORNE, *op. cit.*, p. 5.

sections shown me by Mr Peach. By his advice I searched carefully, on a subsequent visit to the spot, for traces of the piped quartzites and other Cambrian rocks, but failed to observe any. The presence of these pebble beds shows very clearly how great must have been the physical changes which the area had undergone, before sediment so coarse reached districts which had long been the seat of a deposit of the finest grain and the most uniform nature. They are very local in distribution, no trace of the thick beds at Heclabir being found among the yellow sandstones in other areas of Sanday, or indeed anywhere in the district, except on the opposite shore of Eday, where their thickness is quite trivial in comparison.

The yellow beds of Eday and Sanday stretch southward into Shapinshay, where they attain a much greater importance, forming, in fact, the whole thickness of the John o' Groats series in that island. Here the outcrop forms the south-east corner, and is bounded by a line running N.E. from the angle of the bay below the Established Church on the south shore to the Bay of Crook on the east. The underlying flags seem to pass up quite conformably and without any important break into a series of yellow current-bedded sandstones, mixed with numerous thin beds of dark-coloured flags. Along the east side the structure is simplest, the prevalent dips being S.E. and E.S.E., but elsewhere the dips roll greatly, and the beds are evidently being constantly repeated. The yellow sandstones overlying these mixed beds are very pure and massive, and cannot, with any probability, be estimated at less than 400 to 500 feet. Only very rarely is a red-coloured bed of clay to be seen ; but at more than one place there occur belts of flags intercalated between yellow sandstones, and in some places 30 feet in thickness. These flags may be the counterparts in this area of the flagstones which in Eday occupy a similar position, and, like them, they contain the characteristic John o' Groats fossils, one specimen of *Tristichopterus alatus* (Egert.) having been found by me at Store Point in a coarse grey flag. It is among them also that the volcanic rocks * occur which Peach and Horne described as the only evidence of contemporaneous volcanic action in the lower Old Red of Orkney. They consist of a single lava flow, which, though much weathered, is recognisable as an olivine diabase, and is distinctly vesicular at the top surface, while it rests quite conformably on the underlying flag, which is considerably baked and altered.† To their observations I have only a few to add. The interbedded character of the volcanic rock is shown also by the occurrence at its south-western corner of a bed of ash several inches thick immediately overlying it, while in several places thin layers of sprinkled ash can be traced in the overlying flags a few inches apart, and to a distance of 10 feet above the surface of the lava. This shows that though the volcanic activity resulted apparently in only one outflow of lava, it continued for a time to produce occasional showers of ashes, which were spread out over the sea-bottom, and mixed with the sediment accumulating there. At its base the lava contains here and there a bit of an angular baked flag, but its upper surface is vesicular

* Jameson, *Mineralogy of the Scottish Islands*, ii. 235.

† Peach and Horne, *op. cit.*, pp. 9 and 13.

and very irregular, the sandstone filling up all these irregularities quite unaltered and undisturbed in bedding. In several places the lava is 30 feet thick, but in one little creek its top and bottom surfaces were seen in section, and here it was not over 12 feet in thickness. Its greatest development is to the south and east, from which direction it seems to have flowed from a source now, no doubt, concealed by the sea; and this conclusion is strengthened by the occurrence on the same horizon of similar volcanic rocks in the sandstones of Deerness.

The southern termination of this area of John o' Groats beds corresponds very closely with the shores of Inganess Bay. At more than one place in this district the flagstones have yielded *Dipterus macropterus* (Traq.) and *Tristichopterus alatus* (Egert.), and in it occur both types of sediment characteristic of these rocks; but so completely is it occupied by the sea that little certainty can be attained as to its exact geological structure. Along the eastern shores the rocks are yellow sandstones, with many thin beds of dark brittle flag, dipping mostly N.W. at gentle angles. On its western side, again, the red sandstones and marls of Holland Head are underlain by a fine pure yellow sandstone below Berstane House, which at its lower part contains belts of flagstone, and even an occasional red bed. The proximity to the great fault which runs out to sea in Meil Bay disturbs these rocks somewhat, but there can be no doubt that this is the order of the succession, and that, on the whole, these are higher in the series than the yellow sandstones, which on the other side of the bay rest on the flags of the East Mainland anticline. The area must be somewhat disturbed by faults, for on the shore to the southward, at the west corner of Inganess Bay, we find a patch of red marls which belong undoubtedly to the overlying red series. The yellow sandstones of this area bear a close resemblance to those of Shapinshay, from which they differ chiefly in the absence of any interbedded volcanic rocks. They show also that the Shapinshay rocks are merely the basal part of the series, and that overlying the yellow beds in this area, as in Eday, there is a series of red sandstones and marls of considerable thickness.

In Deerness occurs an area of John o' Groats beds which in some respects is the most varied and interesting of any in Orkney. Separated from the previous series by the anticline which brings up the lower flags through the parish of St Andrews, it forms in turn a syncline or basin, of which only the northern half is accessible to observation. The axis of this syncline runs probably E.N.E. from Stembuster on the shore south of Dingieshowie, and on the south side of this axis we have only a very short stretch of sandstones along the shore to just south of the Castle of Claisdie, where they pass down into the grey flags of the parish of Holm. Northwards along the shore the dips sweep round, till at Dingieshowie they are E.S.E.; and E.S.E. and S.E. dips, as already remarked, are far the most prevalent throughout the parish of Deerness. One of the most complete and trustworthy sections is that described by me in a previous paper * as stretching from Dingieshowie to Newark Bay along the south shore, but this is in so far incom-

* *Trans. Roy. Phys. Soc. Edin.*, vol. xiii.

plete that the fault which crosses the isthmus at Dingieshowie cuts out the passage beds underlying the yellow sandstones. These are seen in the shores farther west, at Stembuster, where they consist of thin courses of yellow sandstone with slaty flags between, forming a very gradual transition between the two types of sediment. Above these 'passage beds' lies a series of red marls, with thin yellow and brown sandstones (40 feet) between, recalling in some ways the beds described as occupying a similar position in Eday. The exact point at which the base of this series should be drawn is a matter of some doubt, as among the lowest of them occur beds crowded with *Dipterus valenciennesii* (Sedgw. and Murch.), and containing also *Asterolepis*, sp. nov., but containing no other fishes, an observation due to Mr MAGNUS SPENCE. Traced upwards, the red beds pass gradually into a pure massive yellow sandstone, which forms the high cliff below Tornpike, and is, no doubt, the same as that of Delday's Banks. These lowest beds are exposed also in other parts of the parish, as at Braebuster and the shores to the south of it, where thin grey flags pass gradually up into yellow sandstones. In the north shore of Deerness a very similar series occurs at Halle, and extend thence to near the Covenanters Monument, lying very flat, and forming the extreme N. edge of the syncline; and these rocks must again outcrop in Sandside Bay, between the flags which form its northern side and the yellow sandstones to the south, though here the rock is concealed by the blown sand which occupies the centre of the bay. Just north of the Brough of Deerness the presence of a few red beds beneath the yellow sandstones is well seen in a lofty cliff, and again the same feature is to be observed in the shore below Horraquoy. The yellow sandstones recall, in very many respects, those of Shapinshay. They are of much the same thickness, 400 to 500 feet, and through them lie here and there belts of thin grey calcareous flags, which are the chief source of the John o' Groats fossils of Orkney. They contain *Dipterus macropterus* (Traq.), *Tristichopterus alatus* (Egert.), and *Microbrachius Dicki* (Traq.), the first especially in great abundance, and often in fine preservation; and it is probable that through the low lying centre of the parish these largely replace the yellow sandstone series.

A further point of similarity to the yellow sandstones of Shapinshay is furnished by the presence in these beds of a zone of contemporaneous volcanic rocks of basic composition. These occur rather above the middle of the yellow series, and, as in the district previously described, they are immediately associated with a belt of grey flags intercalated among the sandstones. They consist of both ashes and lavas, and in addition there are several intrusive sheets which, from their composition and general character, are undoubtedly to be ascribed to the same volcanic source.

At the extreme south-east corner of the parish, at the Point of Ayre, a series of volcanic rocks form a narrow belt running W.N.W. in the land, and outcropping on the seashore. The general dip in this quarter is S. and S.S.E., and from Horraquoy southward along the east shore we pass over a gradually ascending section of the lower members of the yellow sandstones. This dip continues to the Point of Ayre, which consists of beds of flagstone, and these, though somewhat faulted, evidently are to be

assigned to the upper part of the yellow beds. From this point westward they strike along the shore, which they form up to the Bay of Newark, where they are covered by blown sand. On the west side of the bay, beds occur striking N.N.E., and evidently let down by means of a dislocation covered by the superficial accumulations in the centre of the bay. The principal mass of volcanic rock at the Point of Ayre forms a narrow area which runs E.S.E. out to sea, and is in breadth about 40 yards. Its base is not seen, and its lower member is a thick bed of dark green volcanic ash, with large spherical bombs up to 2 feet in diameter, vesicular, especially in the centre, and much decomposed. A few bits of baked flag occur in the ash, and it weathers in a markedly spheroidal manner, resembling, in fact, very closely many of the basaltic ash beds around the shores of the Firth of Forth, as at Kinghorn and Elie. In general it shows no trace of bedding, but here and there a few thin irregular lenticles of sand are to be seen, which prove that though rapidly accumulated, it is not the product of a single outburst. A curious feature is the existence in it of flagstone veins. These are very tortuous and irregular, an inch or two in thickness, and filled with a normal, somewhat calcareous flagstone, in which little or no trace of any metamorphism is to be found. They are vertical, and show no sign of bedding or contortion, and are to be regarded as due to the formation of cracks in the thick accumulation of volcanic ash, into which the ordinary sediment of the sea-bottom was washed. At first glance, this bed of agglomerate suggests at once that it is a volcanic neck, and the elongated form of its outcrop would support this explanation. But its junction with the flags to the south is a small fault, and these show none of that alteration which is to be expected in the walls of a volcanic neck. And, moreover, the bed itself is seen in the low cliff to be overlaid by a thin lava, and that again by well-bedded flags. Still, it is in every way probable that an accumulation of this sort was formed in the immediate proximity of a volcanic orifice. The overlying lava is some three feet in greatest thickness, vesicular at its upper surface, the vesicles being large, not markedly elongated, and filled with calcite and other secondary minerals. It is greatly decomposed, but shows little of the spheroidal weathering of the agglomerate, being rather divided by well-marked joints into polygonal vertical columns. Under the microscope it turns out to be an olivine basalt, so greatly decomposed that few of the original minerals remain. At the western corner of the outcrop this lava is seen to be, in turn, overlaid by ordinary flags, which are in nowise altered by the heat of the underlying rock, and contain little or no fragmental volcanic matter. These rocks are bounded to the south, and probably also to the north, by small faults. A few yards to the west of them, what seems to be a quite distinct outflow is exposed in the shore. This is the edge of a small lava flow, three feet in thickness, and thinning out in a few yards to the south, while the flags close over it. It is dark in colour, with large steam cavities in its upper surface, and bears a striking resemblance to the volcanic rock at Haco's Ness, Shapinshay. The sea has removed the overlying rocks, except at the thin edge, where a layer of dark green ashes mixed with sand is seen to immediately overlie the lava, succeeded in turn by a normal unbaked ordinary flag. The lava rests upon a similar

flagstone, and hence cannot be the same as that already described to overlie the thick agglomerate bed, a few yards further to the east.

Among the yellow sandstones, about two miles further to the west along the shore, and about a hundred feet below where they pass into the red sandstones, occurs another belt of contemporaneous volcanic rock. It is associated here, also, with a series of flagstones, and no doubt is on the same level as the rocks just described. In a little bay to the east of the Castle, a bed of dark green ashy sandstones, mostly fine-grained, but with here and there lapilli of a couple of inches in diameter, is to be seen, interbedded with yellow sandstones and flags. It is very similar in character to the ash beds in Shapinshay which overlie the lava; but while these are mostly of very inconsiderable thickness, it is in some places three or four feet thick. No lava is associated with it, and in the sandstones above and below I found no trace of any recurrence of the volcanic activity. In all probability it is the representative, in this section, of the coarse agglomerate already described, which must have greatly thinned out in the intervening distance. The striking feature of this volcanic zone is its very diminutive thickness. Still, the occurrence in Orkney of such a zone is a remarkable confirmation of the opinion expressed by Sir A. GEIKIE, that the "ancient volcano of John o' Groats might be one of a series which might hopefully be sought for among the Orkney Islands." *

Rocks of an intrusive origin occur also in this district, the principal mass being exposed in the locality last mentioned, about 50 yards west of the ashy sandstone. It forms a mass of about 25 feet in thickness, though its base is not exposed, a dark green rock, which is first seen in the shore, and runs out to sea in a series of picturesque stacks and reefs. Its intrusive character is shown by the absence of any amygdaloidal upper surface, and the evidently unconformable junction with the overlying sandstones. Yet these were, so far as I could make out, not markedly altered, though they are so decomposed that this would not be easy to determine. The rock is about 30 feet beneath the ashy sandstone, and in structure is a much weathered diabase, with crystals of plagioclase felspar, augite, and probably olivine, almost entirely decomposed into green chloritic products, which show traces of ophitic structure. Throughout Deerness, in several places, occur masses of volcanic rock so decomposed and so obscured in their geological relations by the surface accumulations that it is not easy to form an opinion as to their true character. They all occur among the yellow sandstones and the flags associated with them. One is seen to the south of the Free Church, and several outcrops are known in the vicinity of the Public School. I am greatly indebted to Mr MAGNUS SPENCE for specimens and observations on these outcrops. From their microscopic structure and the absence of any accompanying tuffs, they are in all probability intrusive sheets. The freshest specimen I obtained was a dark green diabase, with well-marked ophitic structure and pseudomorphs of serpentine after olivine. It came from a deep pit, at one time sunk in a field behind the Public School.†

* Sir A. GEIKIE, "Old Red Sandstone," *Trans. Roy. Soc. Edin.*, vol. xxviii. p. 406.

† The Black Holm of Copinshay consists of an intrusive sheet of olivine diabase about 30 feet thick, enclosing a large mass of baked flag penetrated by numerous veins. This is probably that referred to by JAMESON, *Scottish Islands*, ii. p. 235.

An outcrop of special interest occurs in a field 400 yards west of Smiddybanks. Here, in an old gravel-pit, a face some ten feet high is exposed, now much broken down by weathering. The rock is a coarse red sandy ash, with green spots. In it occur very numerous sandstone and flagstone fragments, some as large as a man's head,—the sandstones baked into quartzites; the flags fused and slaggy on their surfaces, and with their edges rounded. Materials such as these form a considerable proportion of the whole mass. It seems unbedded, or rather the few traces of bedding planes showed a dip discordant with that of the surrounding sandstones. No similar bed crops out along the shore, and the outcrop seems to be limited in area and rudely circular in outline, though, as it occurs in the midst of cultivated land, its exact margins cannot be traced. It is difficult to understand what this is, unless it be regarded as a small volcanic neck, the mixed nature of its fragments being so different from that of the other ash beds, while its position in the centre of the intrusive sheets and lavas and ashes already described renders such a hypothesis, to say the least, highly probable.

There can be no doubt that all these volcanic rocks owe their origin to the same period of volcanic activity. Their situation, almost in the direct line between the Neck of Huna and the lava of Haco's Ness, points to the existence of a north and south fracture or line of weakness, which may be ascribed to the earth movements, which, at the close of the deposition of the Rousay rocks of Orkney, introduced new types of sediment and new forms of life. To the westwards, at any rate, no trace of similar structures has been found. At two subsequent periods volcanic rocks rose to the surface in this district: one series forms the lavas and ash beds of Hoy, described by Sir A. Geikie. These, too, are of basaltic character, but they are separated from those we are at present considering by a great conformity. The others form the trap dykes, which traverse the flagstones mostly in an E.N.E. and W.S.W. direction. But these latter are in no place connected with surface outflows, and differ so widely in structure and composition from the rocks of Deerness and Shapinshay, as undoubtedly to have proceeded from quite distinct sources. They are, in fact, chiefly developed in the West Mainland, and are comparatively few in regions occupied by John o' Groats rocks.

Section 4.—N. and S., through South Ronaldshay. 2 miles = 1 inch.

The only remaining district of the yellow sandstones is the basin of the South Isles. A complete examination of this area I was unable to overtake, but was compelled to confine myself to the islands of South Ronaldshay and Burray, in which they occupy the largest area of any of the South Isles, and very clear sections are to be obtained. Here, also, the underlying yellow series is well developed, and passes down by means of a series of flaggy passage beds into the grey flags, which at the south end of South Ronaldshay contain the Rousay fossils. These passage beds are well seen on the south

shore of Watersound, just east of St Margaret's Hope. At Stews Head they contain a few reddish bands. In South Ronaldshay the yellow series is largely developed, and, with the exception of the district from Widewall to St Margaret's Hope, and thence to Hoxa, they occupy all the areas marked on the map as belonging to John o' Groats beds. A fine section of massive yellow sandstones, with a few flag-beds, is seen extending from Barswick on the west side, north to Herston Head. It is broken by several faults, but there can be no doubt that in thickness it is greater than any other section of the same rocks elsewhere exposed in Orkney. Among these beds no trace of a volcanic zone has yet been discovered, and as yet no John o' Groats fossils have been obtained from any of the South Isles. Their relationships are such, however, as to leave no doubt whatever of their position in the series.

In the district around Melsetter in the island of Hoy, according to PEACH and HORNE, bands of yellow sandstone occur, overlying conformably the flags which form the south end of the island. These resemble greatly the upper Old Red Sandstones of the west end of Hoy, which unconformably overlie the flags. Now, at the west side of Hoy, opposite Graemsay, the upper sandstones rest on flags which are to be correlated with the Orcadian beds of the opposite shores of Stromness. This is clear proof of the great erosion which must have preceded the deposition of the upper Old Red series in Orkney, as time sufficient for the removal of all the Rousay rocks and all the John o' Groats rocks of Orkney must have elapsed before the upper beds were laid down on the upturned edges of the Stromness flags which form the base of the Old Man of Hoy.

The Red Sandstones of the John o' Groats Beds.

The red sandstones of the John o' Groats beds of Orkney have their greatest development in South Ronaldshay, in the extreme south, and in Eday, at the extreme north of the country, while in the intervening districts their thickness is small. In Eday, they form the entire north end of the island, and thence pass down the centre to Sealskerry Bay. Some of the highest elevations along this line have a height of 350 feet, and the least possible estimate of the thickness of the whole series cannot be less than 600 feet. The yellow sandstones of this island are, however, of only slight thickness, and it is possible that the red beds, in fact, replace the yellow, which further south have a much greater development. Red sandstones form also the south-east corner of the island around the point of Veness. To the geologist these beds are somewhat uninteresting. No fossils have been found in them, and they contain no contemporaneous volcanic rocks. The absence of fossils is perhaps due to the fact that there are no beds of close-grained flag suitable for the preservation of organic remains. The beds themselves consist of coarse red sandstones, often in thick beds, alternating with red shales and marls, with sometimes a greenish or greyish shale. In Eday the sandstones greatly preponderate, and in some places are so coarse as to deserve the title of 'grits.' No traces of any chemical deposit, such as rock salt or gypsum, occur any-

where, and the red matter is uniformly disposed through the rock, except where leached out by percolating water, or where aggregated into irregular layers of iron pan.

In Sanday, along the west shore, the beds have a very similar character, but are more friable, owing to the admixture of dark red clay. In Shapinshay red beds practically do not occur, the only representatives of the John o' Groats beds being the yellow sandstones and flags; but on Holland Head red beds again appear, with every peculiarity to be found in those of Eday. Here, again, the beds are mostly massive sandstones, the red shales being of only secondary importance. The total thickness in this section is about 200 feet. In Deerness, red beds form the western shore of Newark Bay, and stretch westwards nearly to the Castle. Here thick coarse sandstones are mixed with green and red marls. The extreme north point of the parish consists of similar rocks, which are let down by a fault running east and west just south of the Mull Head. They have little of the massive uniformity which characterises the beds of Eday, the alternations in the nature of the sediment being comparatively frequent. Red beds form also the cliff above the Scapa Pier, but in the South Isles area, their best exposure is that from Widewall in South Ronaldshay, by Roeberry, to Hoxa, and thence to St Margaret's Hope along the shore. Here the dip is gentle to north and north-west, and the underlying beds of yellow sandstone pass up very gradually into the deep red marls beneath Roeberry House. The thickness of these marls—which contain thin beds of red sandstone—is considerable, and they resemble closely the beds seen in Calf Sound, in Eday, in every respect, except their greater thickness. Similar beds are to be seen below Smiddybanks in St Margaret's Hope. Overlying these there come in massive coarse red sandstones, which occupy the rest of the area up to Hoxa, where they are faulted against the flags of Hoxa Head. The thickness exposed in this section is about 500 feet, and not greatly less than that of Eday, where the yellow sandstones are so insignificant. The whole thickness of the John o' Groats beds of Orkney may thus be put down at about 1000 feet in its greatest development. Red beds occur also in Burray and Hunda, but these present no features of special interest to merit a separate description.

With these red sandstones the long history of the Orcadian Old Red of Orkney comes to a close. A complete change in the nature of the sediment accompanied what must have been considerable changes in the physical conditions of the area. Yet it is, after all, only a reversion to that type of deposit which elsewhere had been the main one for vast periods of time. In the nature of its rocks and in the limited development of volcanic activity, this area had long been a great contrast to the Old Red of Southern Scotland; only at its close do we find a partial resemblance to make its appearance. The red sandstones are the least important part of the Orkney Old Red. Neither in Caithness nor Orkney do we find them conformably overlaid by any other rock. The new conditions which supervened were marked by the precursors of a new fauna, of which the first example is the Asterolepis, a fish so characteristic of the upper Old Red of the southern shores of the Moray Firth. But before that fauna was to attain its greatest development great changes in the physical geography of Scotland had to take

place, and vast periods of time to elapse. Before the deposit of the upper Old Red of Hoy, much of the Orcadian Old Red had been stripped from the surface of the Orkneys, and very considerable dislocations had modified entirely the old physiography and structure of the country.

III.—*Comparison with the Old Red of other Districts.*

Such being in its main features the structure of the Orkneys, and the subdivisions which can be established by the distribution of the fossils, it remains to be considered how far these conclusions can be applied to other districts in which rocks of like age and similar fossils occur.

The John o' Groats Beds and the Eday Sandstones.

As regards the uppermost beds, the inquiry is a simple one. Rocks containing the same fossils occur in only one locality—the north-eastern angle of Caithness; and here their lithological characters so strikingly resemble those of the Orkney beds that no difficulty whatever can be felt in accepting their zonal identity. The John o' Groats beds of Caithness are, then, to be correlated with the Eday, Deerness, and South Ronaldshay sandstones of Orkney. Sir A. GEIKIE gives a list of the fossils which have been found in this series in Caithness.* He enumerates, in addition to the three type fossils, *Acanthodes Peachi* (Eg.) and *Glyptolepis leptopterus* (Ag.), neither of which is known to be present in the similar beds of Orkney. It is remarkable how in both counties the fishes characteristic of the lower rocks have been superseded by new types so completely that almost no trace of their persistence is to be obtained. The uppermost zone of the Orcadian Old Red is thus a well characterised one, and may be designated, from the locality in which alone it was known to occur for many years, The John o' Groats Sandstones (zone of *Tristichopterus alatus*, Egert.).

The Thurso and Rousay Beds.

For the representatives elsewhere in Scotland of the lower zones, we must look to two localities, to Cromarty, from which HUGH MILLER and AGASSIZ early in the century furnished a list of fossils, and to Caithness, where, since the time of HUGH MILLER and ROBERT DICK, much work has been done in the palæontology of the Old Red. The earlier work has subsequently been subjected to thorough revision, and a wealth of new material been brought to light by Dr TRAQUAIR, to whose papers I am greatly indebted, and on whose published statements I shall rely in comparing the lists of fossils from each locality. In his paper, "Achanarras Revisited" (1894),† he has briefly stated the

* Sir A. GEIKIE, *Old Red Sandstone*, p. 404.
† TRAQUAIR, *Proc. Roy. Phys. Soc. Edin.*, vol. xii., 1894.

results of a comparison of lists of fossils from Caithness, Orkney, and Cromarty, and the result is a division of the known fossils into three groups. One is that we have already considered—the John o' Groats group. The second contains a series of fossils which occur together only in the neighbourhood of Thurso. The list is as follows :—

Homacanthus borealis (Traq.).
Rhadinacanthus longispinus (Ag.).
Mesacanthus Peachi (Egert.).
Cheiracanthus, sp. (perhaps 2 sp.).
Coccosteus decipiens (Ag.).
Coccosteus minor (H. Miller).
Homosteus Milleri (Traq.).
Dipterus valenciennesii (Sedgw. and Murch.).
Glyptolepis paucidens (Ag.).
Thursius macrolepidotus (Sedgw. and Murch.).
Thursius pholidotus (Traq.).
Osteolepis microlepidotus (Pander.).
(*Scales, doubtfully resembling those of* Gyroptychius).

It will be observed that this list contains the type fossils of the Rousay series of Orkney, *Coccosteus minor* (H. Miller) and *Thursius pholidotus* (Traq.); and when we compare it with the list of the fossils I have found in those rocks, we find that the following species occur in both :—

Coccosteus minor (H. Miller).
Thursius pholidotus (Traq.).
Dipterus valenciennesii (Sedgw. and Murch.).
Glyptolepis paucidens (Ag.).
Cheiracanthus, sp.
Coccosteus decipiens (Ag.).
Homosteus Milleri (Traq.).

With the exception of the first two, these are all contained in the list of fossils which occur throughout the whole thickness of the Orkney flagstones. In Orkney occurs one species not yet found in Caithness, *Asterolepis*, sp. nov., which, considering that it is a fossil of limited range, and confined to a few beds of rock, is an exception of no great importance; and two others present in Caithness, but not known from the vicinity of Thurso, *Osteolepis macrolepidotus* (Ag.) and *Diplopterus Agassizi*. Of these, the latter is one of the rarest of Caithness species, while in Orkney it is quite common, especially in the quarries of Sandwick and Stromness. From the Rousay beds of Orkney I have seen only one satisfactory specimen. It is probable that we have here a case of local distribution, and that the absence of this fossil from the rocks around Thurso is due, not to adverse conditions of preservation, but that rather it was from the first a species characteristic of the more northern area, and hence more likely to persist there, and occur on a higher horizon. On the other hand, we have a number of forms known

to occur near Thurso, but not found as yet in the Rousay beds of Orkney. These are :—

Homacanthus borealis (Traq.).
Rhadinacanthus longispinus, (Ag.).
Mesacanthus Peachi (Egert.).
Thursius macrolepidotus (Sedgw. and Murch.).
Osteolepis microlepidotus (Pander.).

Of these, the first is a rare fossil, and only described for the first time in 1892.* The second cannot be regarded as very abundant, seeing that the British Museum Catalogue (1891) does not enumerate it as a Caithness species. The third has not, so far, been mentioned in the literature of Orcadian geology, though Dr TRAQUAIR, I believe, has obtained a species of Mesacanthus from Orkney this last summer. That these three rarities should be known from the carefully examined rocks around Thurso, and not as yet from the Rousay beds of Orkney, to which attention has only lately been directed, cannot be regarded as a strong argument against the theory that the one series is the northern representative of the other. The two remaining fossils are of more importance, seeing that they are regarded by Dr TRAQUAIR as typical of the Thurso rocks, and confined to them. One of these, *Osteolepis microlepidotus* (Pander.), is very characteristic of them, and abundant in some of the beds ; but I have, at many different times, examined collections of Orcadian fossils, and carefully searched the rocks for this species, without ever obtaining a specimen which Dr TRAQUAIR would admit belonged to it. No doubt it has figured more than once in lists of fossils from Orkney, but the identification is at present more than doubtful. It is possible that we have here a case of local distribution the converse of that of Diplopterus, but at any rate the discrepancy is one which cannot be overlooked, and it is to be hoped that further search in Orkney will bring this fish to light. *Thursius macrolepidotus* (Ag.), it may also be anticipated, will turn up in the Rousay beds, or at any rate its absence is not very remarkable when we remember that only one satisfactory specimen of the other species of the same genus has yet been discovered. Yet that, in that case, in the same quarry, two species which, according to Dr TRAQUAIR, are typical of the Thurso rocks, should have been found together for the first time in Orkney, is a surprising confirmation of the views he enunciated in 1894, that they are type species of a special subdivision of the Orcadian Old Red ; and that their distribution in Orkney, so far as yet known, is in complete accordance with this supposition, has already been proved to be the case. They occur always on practically the same horizon, and in the lowest beds they have never yet been found.

No other locality for these two fossils is at present known, and from the district in which they have been longest and most thoroughly investigated they may be named the Thurso Beds, or the Zone of *Coccosteus minor* (Hugh Miller) and *Thursius pholidotus* (Traq.).

* *Trans. Geol. Soc. Edin.*, 1892.

The Cromarty, Achanarras, and Stromness Beds.

The third group of fossils recognised by Dr TRAQUAIR is that which HUGH MILLER first described from Cromarty, and he himself, on several occasions, from Achanarras (Caithness), and which was long believed to be the only one present in the Orkneys.

The following is a list of the fossils of Cromarty, Achanarras, and the Stromness beds of Orkney.—

Palæo spondylus Gunni,		A.	
Diplacanthus striatus (Ag.),	C.	A.	O.
„ *tenuistriatus* (Traq.),	C.		
Rhadinacanthus longispinus (Traq.),	C.	A.	O.
Mesacanthus pusillus (Traq.),	C.	A.?	O.?
Cheiracanthus Murchisoni (Ag.),	C.	A.	O.
„ *latus* (Egert.),	C.		
„ *grandispinus* (M'Coy),			O.
Pterichthys Milleri (Ag.),	C.	A.	O.
„ *productus* (Ag.),	C.	A.	O.
„ *oblongus* (Ag.),	C.	A.	
Dipterus valenciennisii, (Sedgw. and Murch.),	C.	A.	O.
Coccosteus decipiens (Ag.),	C.	A.	O.
Homosteus Milleri (Traq.),	C.	A.	O.
Glyptolepis paucidens (Ag.),		A.	O.
„ *leptopterus* (Ag.),	C.		O.
Gyroptychius microlepidotus (Ag.),	C.	?	O.
Diplopterus Agassizi (Traill),	C.	A.	O.
Osteolepis macrolepidota (Ag.),	C.	A.	O.
Cheirolepis Trailli (Ag.),	C.	A.	O. *

A glance will show the very complete accordance of these lists. All the more frequently occurring fishes are common to all the localities, except possibly *Glyptolepis paucidens* (Ag.), which in the Cromarty district is replaced by the closely allied *Glyptolepis leptopterus* (Ag.). *Gyroptychius microlepidotus* (Ag.) seems to be absent from the Caithness area. The other fishes found in one area only are all rare fossils.

If, now, we examine the list to ascertain which fossils are confined to these areas, we find that—

Palæospondylus Gunni (Traq.)
Diplacanthus, 2 sp.
Pterychthys, 3 sp.
Cheirolepis Trailli (Ag.)

and possibly

Gyroptychius microlepidotus (Ag.)

are not known to occur elsewhere.

* This list has been compiled from—TRAQUAIR, "Fossil Vertebrates of the Moray Firth"; TRAQUAIR, "Achanarras Revisited"; A. S. WOODWARD, British Museum Catalogue of Fossil Fishes

To these Dr TRAQUAIR adds two—*Osteolepis macrolepidotus* (Ag.), which certainly occurs in the East Mainland of Orkney, and *Diplopterus Agassizi* (Traill), which, he says, he has not been able to establish with certainty as a member of the Thurso group. If we except the rare *Palæospondylus Gunni* (Traq.), which is known only from Achanarras, we have three genera and six species which, so far as our present knowledge of the distribution of the fossil fishes of the Scottish Old Red Sandstone goes, may serve as type fossils for this group of rocks; and these, it will be remembered, are the genera which I found in Orkney to characterise the Stromness beds; and we may regard it as established that this is a distinct zone of the Orcadian Old Red Sandstone, of which the representatives are the sandstones of Cromarty, Lethen, Gamrie, Clunie, and Tynet, the flagstones of Achanarras in Caithness, and the Stromness beds of the Orkneys.

In conclusion, I wish to acknowledge my indebtedness to those who have assisted me in this work—to Professor JAMES GEIKIE, D.C.L., LL.D., F.R.S., without whose encouragement and advice it would never have been undertaken; to Dr R. H. TRAQUAIR, LL.D., F.R.S., who has determined for me all the more important specimens collected, and has kindly undertaken the description of the new material which turned up in the course of the investigation; to Messrs BENJAMIN PEACH, F.R.S., and JOHN HORNE, F.G.S., of the Geological Survey of Scotland, who have at all times placed at my service their great knowledge of field work, and their intimate acquaintance with the geology of the district. Mr JAMES W. CURSITER, F.S.A. Scot., of Kirkwall, kindly placed at my disposal his fine library of books relating to the county, and his collection of Orkney fossils; Mr THOMAS M'CRIE, of Kirkwall, allowed me also to examine his collection; and Mr MAGNUS SPENCE, of Deerness, gave me most valuable assistance in the field work in that district and elsewhere.

PLATE I.

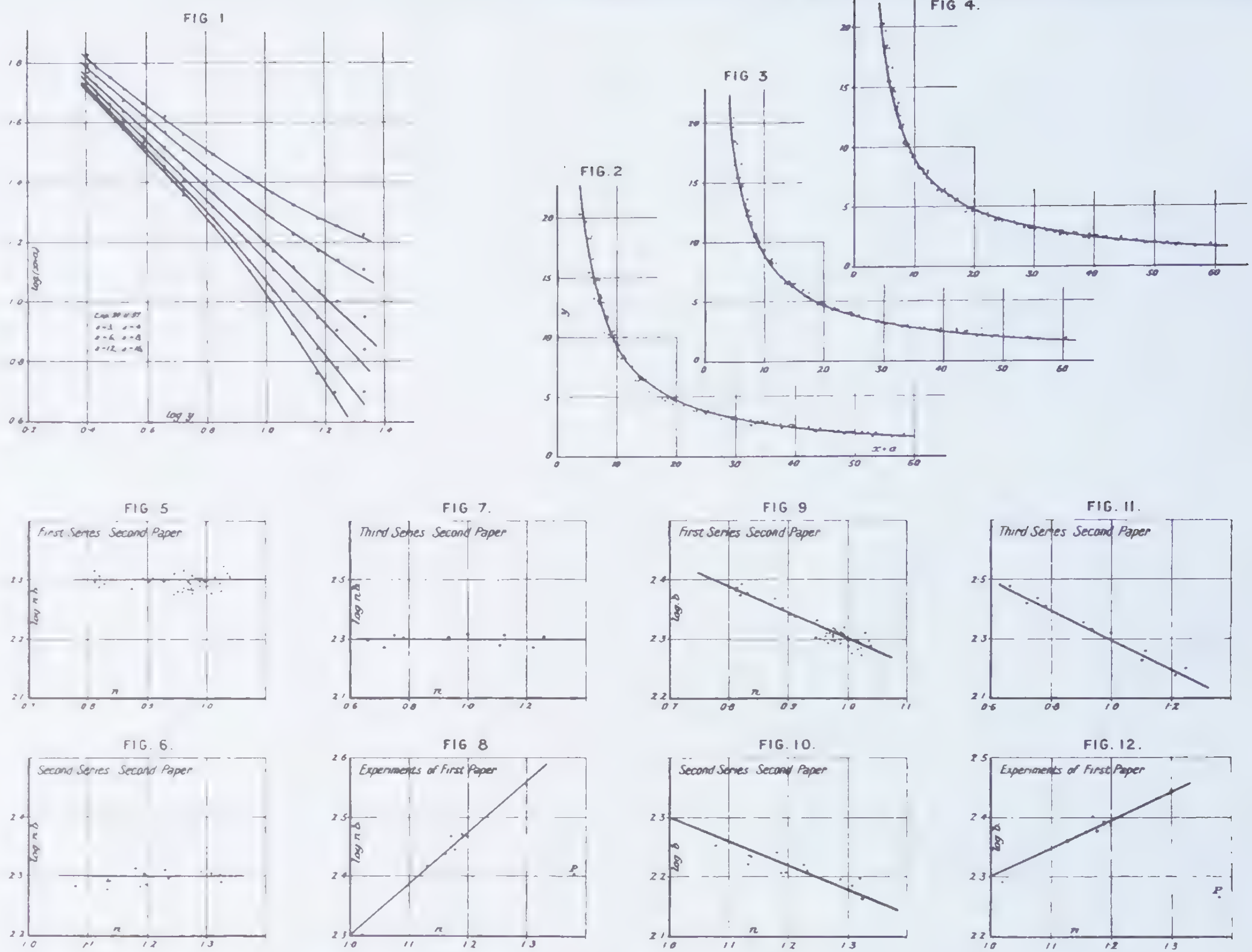


FIG 1
Log y
FIG. 2
FIG 3
FIG 4.
x + a
FIG 5
First Series Second Paper
FIG 7.
Third Series Second Paper
FIG 9
First Series Second Paper
FIG. 11.
Third Series Second Paper
FIG. 6.
Second Series Second Paper
FIG 8
Experiments of First Paper
FIG. 10.
Second Series Second Paper
FIG. 12.
Experiments of First Paper

FIG 14

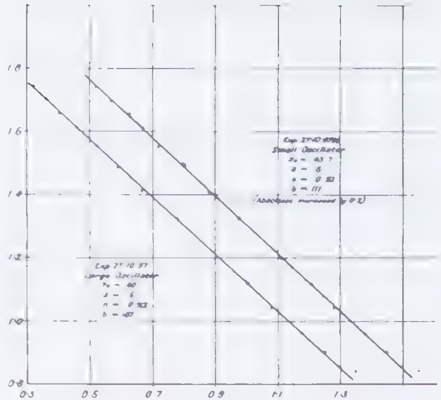


FIG 13.

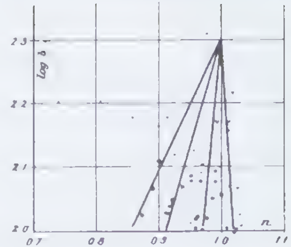


FIG. 15.

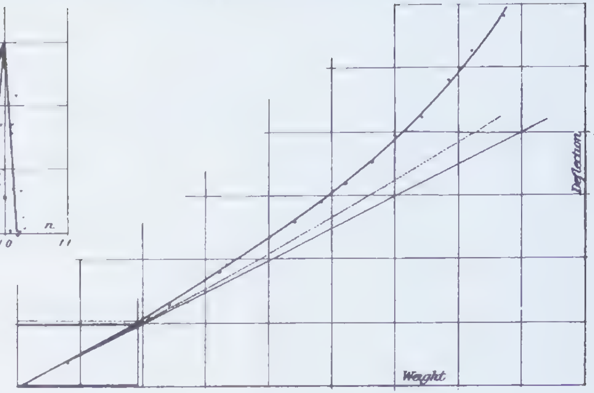
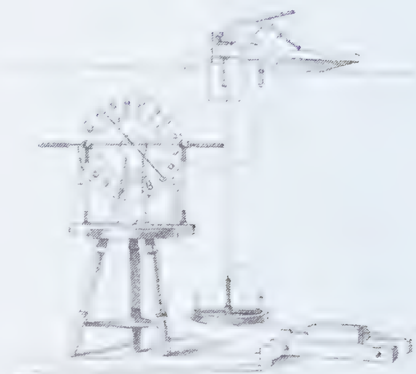


FIG. 16

FIG. 17.



XIV.—*On Torsional Oscillations of Wires.* By Dr W. PEDDIE. (With Two Plates.)

(Read 20th June 1898.)

This paper is in continuation of two others, on the same subject, previously communicated to the Society. In the First Paper (*Philosophical Magazine*, July 1894) it was shown that the formula

$$y^n(x+a) = b,$$

where n, a, and b are constants in any one experiment, represents with accuracy the relation between y, the range of oscillation, and x, the number of oscillations which have taken place since torsion was first applied and the wire was left to itself, so that the oscillations gradually diminished. The apparatus employed, and the method of observation used, were identical with those described in the Second Paper above referred to. The wire which was experimented upon was the same as that used on the previous occasions. Its length, as given in the First and Second Papers, was 89·1 cm. A measurement made on the date 19.10.1897, in the course of the last series of experiments described in the present paper, showed that the length had become 89·3 cm. This increase was doubtless due to the fact that the heavy lead oscillator had been left attached to the wire during the whole of the intervening period. On the date given, it was also found that, with the same oscillator as was used in the experiments first described, ten oscillations were performed in 81 seconds, when the range was large, while 79 seconds were occupied when the range was small. This observation verified the result stated in the First Paper, that the period slightly increases as the range increases. It also showed that the wire was practically in the same condition as it was at first, in so far as elastic qualities are concerned; for the corresponding periods were only slightly less in earlier experiments, the difference being largely accounted for by the slight increase of length of the wire.

In the First Paper, the above equation was also deduced as an approximation, from the assumption that the defect of the potential energy of the system, at any given distortion, from the value which it would have had in accordance with Hooke's Law, was proportional to a power of the distortion. It was pointed out that the value of n seemed to approximate to zero when the range of oscillation was very small; and that, when n becomes zero, the equation changes form and becomes the well-known exponential equation, which was first proved by Lord KELVIN to hold when the oscillations are small.

An improved method of calculating the values of the quantities n, a, and b was

described in the Second Paper. That method was employed in the calculations to be given subsequently. Since

$$n \log y + \log (x+a) = \log b,$$

if $\log (x+a)$ be plotted against $\log y$, the corresponding points lie on a straight line which intersects the axis along which $\log y$ is measured at an angle whose tangent is n—provided that the proper value of a is used. The value of b can then be obtained. If a wrong value of a be used, the points will not lie on a straight line. If too large a value of a is taken, the curve on which they lie is convex towards the origin; if too small a value is taken, the curve is concave towards the origin. In this way the true values of the constants are obtained in any experiment. Fig. 1 illustrates the method.

First Series of Experiments.

Previous attempts to separate the effects of the magnitude of the initial oscillation and of fatigue upon the values of the quantities n and b had not been successful. An attempt was therefore made to eliminate entirely the effect of magnitude of range by inducing very great fatigue in the wire. Before this was done a single experiment was made on the date 8.6.96, the wire having practically not been oscillated since the conclusion, on the date 24.12.95, of the third series of experiments described in the Second Paper. After the date 8.6.96, the wire was oscillated three or four times per week, by from 20 to 40 complete oscillations of large magnitude, until the date 10.7.96, when 150 large oscillations were given. Then, on the dates 14.7.96 and 15.7.96, respectively, 40 and 5 large oscillations were given. No readings of the decrease of range with increase of number of oscillations, when the wire was left to itself so that the oscillations died away, were taken on any of these occasions—the object being merely to induce excessive fatigue as a permanent condition in the wire. Such readings were taken on ten succeeding occasions. On each occasion the wire received 25 complete large oscillations, and was then brought to rest before being started anew in oscillation, when the readings were commenced.

Table I. gives the results obtained, the quantities a, n, and b being calculated in the manner already referred to. The magnitude of the initial range y_0 varied greatly in different experiments. The table also includes the results of the experiment made on the date 8.6.96. These show that the wire was practically in the same condition that it had been left in at the conclusion of the previous experiments. On the other hand, the results of the experiments made under conditions of great fatigue of the wire show a marked change in the state of the wire. The value of the product nb has attained a practically constant value, about equal to one-half of its previous value. The values of n and b are practically constant also, though the initial range varies greatly. The double sets of results given under two dates correspond to slightly different inclinations of the line in the diagram used to determine n and b.

Fig. 2 shows the result of taking $n = 1{\cdot}02$, $b = 98$, and choosing a for each experiment, so as to make the points taken from observation in each experiment lie, as far as possible, on a single curve. Ordinates (y) represent range of oscillation, and abscissæ represent number of oscillations (x) plus a. The diagram shows that an improvement might be made by taking n larger, the product nb being still kept equal to 100. The result is given in fig. 3, the value of n being 1·03, while that of b is 97. It appears from that figure that an increase of b would introduce further improvement. The result of making $n = 1{\cdot}03$ and $b = 100$ is shown in fig. 4. The closeness with which the points lie on the curve is quite sufficient to justify the adoption of the general equation

$$y^{1\cdot 03}(x + a) = 100$$

to represent the results of the whole series of experiments. As a rule, the points which correspond to the first readings taken after the oscillations were started in each experiment are those which lie furthest off the curve. If the first readings were as accurate as the others we should have

$$a = 100\, y_0^{-1\cdot 03}$$

where y_0 is the first reading. It is desirable to determine whether or not a slight modification of this expression for a will apply when the actually observed values of y_0 are used. The data below show that this is the case. The first row gives the observed values of y_0. The second gives the values of a, which were employed in order to make the points agree well with the curve shown in fig. 4. The third row gives the values of a, calculated by the above expression; the fourth gives the values of the differences between the observed and the calculated values of a; and the fifth gives the values of a, if we assume 1·4 to be the true value of that difference, and calculate a from the expression

$$a = 1{\cdot}4 + 100\, y_0^{-1\cdot 03}$$

The initial reading, 8·05, taken on the date 22.7.96, totally disagrees with the second, third, and subsequent readings, and seems to have been a mistake. A value 7·5 is much more in accordance with the others.

7·5	16·5	20·3	26·2	29	30	31·5	32·5	35·1	45·2
14·2	7	6	5	4·5	4·4	4	4	4	3·8
12·6	5·6	4·5	3·7	3·1	3·0	2·86	2·77	2·56	2·36
1·6	1·4	1·5	1·3	1·4	1·4	1·1	1·2	1·4	1·4
14·0	7	5·9	5·1	4·5	4·4	4·3	4·2	4·0	3·8

The numbers in the last row agree sufficiently well with those in the second to justify the adoption of the general formula

$$y^{1\cdot\text{[illegible]}}(x+1\cdot4+y_0^{-1\cdot\text{[illegible]}})=100$$

for the representation of the results of the whole series of experiments made under the condition of equal large fatigue.

Table II. contains a comparison, in the case of each experiment, of the results of observation with those of calculation. The middle column in each case contains the observed values of y, when x has successively the values 1, 2, 3, 5, 7, 10, 15, 20, 25, 30, 35, 40, 45, and 50. The numbers in the left hand column are those calculated for the same values of x, with the values of a, n, and b, given in Table I.; those in the right hand column are the corresponding values obtained by means of the general formula just given. The latter have been kindly calculated for me by Mr W. THOMSON, formerly Donald Fraser bursar in the Physical Laboratory. In practically all cases, excepting the one in which the initial range had its largest value, the numbers in the third column agree at least as well with those in the second as do those in the first.

Discussion of the Initial Ranges in Previous Experiments.

If we take the data for the experiments detailed in Tables IV. and V. of the Second Paper (*Trans. R.S.E.*, 1896), and calculate from them, for these experiments, the values of p in the expression

$$y^n(x+p+b\,y_0^{-n})=b,$$

we get interesting evidence of the effect of magnitude of initial range and of fatigue upon the value of p. The results are given in Table III. In the first set, the initial range, y_0, is fairly constant. The numbers in the column headed N give the number of large oscillations to which the wire was subjected before readings were taken. These numbers, therefore, to some extent, indicate the amount of fatigue. They do not do so entirely, since the effect of previous fatigue persists to some extent from day to day. This is indicated by the smaller values of p on succeeding dates, when N had a given value. When fatigue is small, p bears a large ratio to a; when fatigue is great, p bears a small ratio to a.

In the second set, fatigue was practically constant while the initial range varied between wide limits. As was to be expected, p practially vanishes in comparison with a when the initial range is very small, so that the curve $y^n(x+a)=b$ is very flat.

Re-calculation of Data in Table I. of the First Paper.

The values of n, a, and b, given in Table I. of the First Paper (*Philosophical Mag-*

azine, July 1894), were obtained by superposing the experimental curves upon sets of curves of the required form, and choosing the one which gave best correspondence. A re-calculation of the values, by the method now employed, was made, in order to get a strict comparison of the earlier results with those more recently obtained. Table IV. contains the values so found. The columns headed n', a', b' contain the values of the quantities n, a, and b given in the First Paper. The column headed b'' contains the values of b, calculated by the present method, with the old unit for y (0·364 times the new unit used in the Second Paper and the present paper). The columns headed n, a, and b give the values found by the present method in the new unit. The values of n and a are independent of the y-unit. Table VI. is, in part, a reproduction of Table II. of the First Paper. Values of y are given in the top row, and corresponding values of $x+a'$ are given in sets of three rows, each set corresponding to one experiment. The middle row of each set gives the experimentally observed values of $x+a'$; the upper row of each gives the values of $x+a'$ calculated by means of the values of n', a', and b', given in Table IV.; and the lower row gives the values of $x+a'$ calculated by means of the values of n, a', and b'', given in that table. The new values are, on the whole, just as suitable as the old values, and are accordingly used in the subsequent discussion.

Relations between n *and* b.

It was pointed out, in the Second Paper, that, throughout the three series of experiments therein described, the value of the product nb was, within possible experimental errors, constant. The basis for this statement is exhibited graphically in figs. 5, 6, 7. In these figures the values of log nb are plotted as ordinates against the values of n as abscissæ. The average values of log nb was in each case taken to be 2·3. By means of the re-calculated values of n and b for the series described in the First Paper, a similar diagram (fig. 8) was obtained for that series. With the single exception of experiment P, all the points group very well about a straight line having a positive slope. This implies the existence of a *Critical Angle* (see Second Paper) throughout the series of experiments described in the First Paper; so that, by a proper choice of the y-unit, the value of nb might have been made constant in that series also. For the equation

$$ny^n(x+a) = nb$$

may be written in the form

$$ny'^n(x+a) = nb\left(\frac{1}{k}\right)^n$$

by making $ky' = y$, *i.e.*, by taking as the unit a quantity k times greater than the

unit in terms of which y was measured. And, if we denote the quantity on the right hand side of the equation by B, we get

$$\log(nb) = \log B + n \log k,$$

which, when k is constant, is the linear relation above referred to.

But the value of n is such, throughout each series of experiments, that it is impossible to determine whether that relation, or a linear relation between log b and n, is the more accurate. If one were strictly accurate in a given series, the other cannot be so simultaneously. Yet the possible variations in the determined values of n and b, for any experiment in a given series, are such that either relation may be regarded as practically correct. The results for the latter are exhibited graphically in figs. 9, 10, 11, and 12.

Just as the maintenance of a linear relation between log nb and n, in a given series, implies the existence, throughout that series, of a Critical Angle at which the loss of energy per oscillation is independent of n; so the maintenance of a linear relation between log b and n, in a given series, implies the existence, throughout that series, of an angle at which the loss of energy per oscillation varies inversely as n. For the equation

$$y^n(x+a) = b$$

may be put into the form

$$y'^n(x+a) = b\left(\frac{1}{k'}\right)^n$$

by taking as the y-unit a quantity k' times greater than the unit in terms of which y was measured. And k' can always be chosen so that the right hand side of the equation has a given constant value, β say. We then have

$$\log b = \log \beta + n \log k',$$

which, when k' is constant, is the second linear relation. Also

$$\frac{dy'}{dx} = -\frac{1}{n\beta} y'^{\,n+1}.$$

Hence, when y' is unity, *i.e.*, when $y = k'$, dy'/dx and $y'dy'/dx$ vary inversely as n, the latter quantity is practically proportional to the loss of energy per oscillation. For convenience of reference we may call k' the *Inverse Angle.*

Existence of an Oscillation Constant.

As we have just seen, we can always choose a unit k'', which will make the relation between y and x take the form

$$y^n(x+a) = A,$$

where A is an absolute constant. We may call this quantity, k'', the *Unifying Angle,*

since it gives the value of a y-unit, which, in each case, makes b take the absolutely constant value A. Its magnitude is given by the relation

$$k'' = \left(\frac{b}{A}\right)^{\frac{1}{n}}.$$

If a simple expression such as this, connecting the Unifying Angle with the observed quantities n and b in each experiment, did not exist, we could not regard that angle as a quantity possessing any physical importance whatsoever. Indeed, we could not regard it as such unless the quantity A is found by experiment to correspond to some physical constant.

A glance at figs. 5–12 makes it apparent that, in each series of experiments, the lines representing the linear relations already discussed, pass with great accuracy through the point corresponding to $n = 1$, $\log b = 2{\cdot}3$. The value $b = 200$ is therefore of distinct physical importance in all the series. By giving A this value, and eliminating B and β from the linear equations, we get

$$k' = \left(\frac{b}{A}\right)^{\frac{1}{n-1}},$$

and

$$k = \left(\frac{nb}{A}\right)^{\frac{1}{n-1}}.$$

Thus the Inverse and Critical Angles have also simple expressions in terms of b and n.

The quantity A is an *Oscillation Constant* which depends essentially upon the material of which the wire is made. Further evidence regarding its constancy will be given immediately.

Second Series of Experiments.

In order to obtain further evidence on points already referred to, a second series of experiments, commencing on the date 14.10.97, was made. Between that date and the date 30.7.96, on which the first series was concluded, the wire had not been oscillated except on a few occasions in November 1896, and again in March 1897. The results are given in Table V.

At the end of the first experiment it was found that 36½ full oscillations took place in 5 minutes when the oscillations were large, while 37 took place in the same time when the oscillations were small. At the end of the experiment dated 15.11.97 (1), 38 half oscillations took place in 2½ minutes when the oscillations were small.

The values of a, n, and b, which are obtained when y_0 is very small, are extremely uncertain; yet there is no doubt that the value of n is considerably less than unity under that condition, and that the value of b is large.

In the earlier experiments of this series there is evidence that the wire had recovered to a slight extent from the state of fatigue induced in the first series. But

the subjection of the wire to a comparatively small number of full oscillations (given in brackets in Table V.) before an experiment was made, reduced n and b to values like those which were obtained in the first series. This was the case even when y_0 was comparatively small—see experiment 12.11.97 (1).

The most important object of the present series of experiments was to determine whether or not, under different initial conditions, points representing simultaneous values of log b and n still practically lay upon straight lines passing through the point (2·3, 1). This was found to be the case. At first the slope of the line was found to be positive, as it was in the experiments described in the First Paper. The slope of the line increased, under increased fatigue, until it became practically vertical. The wire was very sensitive to variations of fatigue, whether due to magnitude of initial range or to repeated oscillations. Increased fatigue causes an increase of n and a diminution of b: see, for example, experiments 11.11.97 (1) and (2); experiments 16.11.97 (1) and (2); and experiments 17.11.97 (1), (2) and (3).

Fig. 13 represents a number of the results graphically. The group of three points marked thus ⊙ corresponds to the first three experiments. The group marked × corresponds to the next nine experiments; those marked ⊡ correspond to the next ten; those marked v correspond to succeeding experiments in which fatigue was large; and those marked by single points correspond to some of the experiments in which fatigue was small. It is evident that the various groups throughout each of which fatigue was fairly constant are collected in the neighbourhood of straight lines passing through the point (2·3, 1). Variations may be due to slight differences of condition as to fatigue or to the fact that a is always chosen as a whole number, while the most suitable value may lie between two consecutive whole numbers. If, in any case in which a is small, an error of unity were made in the value of a, the corresponding value of n would change by 0·06 or 0·07, while the value of log b would only change by about 0·015 or 0·02. As an error of unity, when a is small, is impossible, it is evident that the grouping of the points round the lines cannot be regarded as accidental.

It therefore appears that the *Oscillation Constant*, A, *is truly a constant throughout all the treatment to which the wire has been subjected.*

Recovery from Fatigue.

The data given, Table V., show that the wire recovers partially from the effect of fatigue with considerable rapidity. Compare, for example, the data for the experiments 16.11.97 (2) and 25.11.97. This is most marked in the case of small oscillations—see 12.11.97 (1) and 17.11.97 (1), the former experiment being made immediately after heavy fatigue, while the latter was made one day after heavy fatigue.

There is another fact which may possibly bear on the question. In some of the

curves obtained by plotting log $(x+a)$ against log y, when the initial oscillation is small, though a straight line passes with considerable accuracy in the neighbourhood of the points, leaving as many points on one side as on the other on the average, yet almost absolute accuracy would be obtained by drawing *two* lines meeting at a very slight inclination—the smaller value of n corresponding to the smaller oscillations. The crossing point of these lines may possibly indicate an angle of torsion, such that molecular groups which break at a less angle have recovered from fatigue, while those which break at a greater angle have not yet recovered from fatigue. I first observed this in the experiment 17.11.97 (1), but it was found subsequently in other experiments, and had also occurred in previous experiments, as detailed below.

It first appeared in the experiment 3.11.97 (2) with $y_0=12{\cdot}8$, and it appears slightly also in the succeeding experiment 4.11.97 (1) with $y_0=20{\cdot}7$. It occurred also in the experiment 9.11.97. In the case of the three experiments of date 10.11.97, it appeared markedly in the first, very slightly, if at all, in the second, and not at all in the third—each experiment apparently aiding in its obliteration. The initial angles in these cases were 13·1, 11·0, and 11·2 respectively. It could not be said to be evident in the experiment 11.11.97 (2), $y_0=9{\cdot}3$, which followed immediately after the experiment 11.11.97 (1), $y_0=35{\cdot}6$; and it did not appear in the experiment 12.11.97 (1), $y_0=9{\cdot}4$, which was immediately preceded by 40 large oscillations. In the experiment 15.11.97 (1), $y_0=8{\cdot}6$, made after the wire had remained at rest for three days, it again appeared markedly, the point of junction of the two lines corresponding to an angle about one and a half times as large as that indicated in the experiment 10.11.97 (1). It could not be observed in the experiment 16.11.97 (1), which followed a large oscillation on the preceding day, though it would appear if a smaller value of a were chosen. But a smaller value of a would increase the value of n, and it is to be noticed that the values of n and b, found for that experiment and the preceding one, are abnormally large (see 18.11.97 (1)). As already mentioned, the peculiarity appears in the experiment 17.11.97 (1), $y_0=14{\cdot}3$, the wire having been considerably fatigued on the preceding day. It did not appear in the subsequent experiments on that date. It was evident in the experiment 18.11.97 (1), $y_0=9{\cdot}8$. In the succeeding experiment on the same date, $y_0=10$, it was also apparent, but the joining point of the lines occurred at a smaller angle. It could not be said to appear in any of the succeeding experiments. In these the initial range was very small, or very large; or, the initial range being of intermediate size, the experiments were made when the wire had been only slightly oscillated for some days, in which case the joining point might be expected to occur at smaller angles than those which were observed.

The phenomenon, although not very readily observed, occurs with such persistency that I scarcely think that it can be due to accidental causes. The facts that the joining point occurs at a larger angle when fatigue is small than when it is large, and that repetition of an experiment with small initial range makes the joining point pass to smaller angles, seem to indicate that there is a fairly sharply-marked limiting angle,

below which recovery from fatigue has proceeded to a greater extent than it has for larger angles of distortion.

Zero Effect of Period of Oscillation.

In order to determine whether or not the period of oscillation had any influence on the values of n and b, on the date 27.10.97, the large oscillator was replaced by the oscillator of smaller moment of inertia, which was used in the experiments described in the first paper. The results are given in fig. 14. A comparison of the results given in Table V., for the experiment 27.10.97 (2), with the results for previous experiments with the large oscillator, *e.g.*, with the results for the experiment 20.10.97, shows that no change by halving the period. With such speeds of oscillation we must therefore regard the results as independent of "after-action."

Law of Oscillation.

We have already found that the period of a complete oscillation is very nearly constant, being slightly greater for large oscillations than for small oscillations. Some additions were made to the apparatus in order to make possible determinations of the times of outward and inward motions over a given range. Fig. 17 shows the details. The torsion head, to which the upper end of the vertical wire is attached, is seen at the top of the diagram. The horizontal lead ring is seen attached to the lower end of the wire. A Wimshurst machine is seen on the left side of the wire. A vertical glass tube is seen at one extremity of a diameter of the lead ring. Its lower end is drawn to a fine point, and it is filled with a coloured liquid. A similar tube is placed at the other end of the diameter of the ring to secure symmetry in the oscillator. The liquid in the tube is placed, by means of a copper wire, visible in the diagram, in electric connection with the lead ring; and a copper wire also connects the torsion head (which is insulated by means of blocks of paraffin from the support to which it is clamped) to one pole of the Wimshurst machine. When the machine is worked, the liquid is driven out of the tube in a fine jet. On the right hand side of the diagram, at a lower level than the lead ring, are seen massive iron blocks, between which is clamped a horizontal steel wire, which is weighted at its outer end in order to give it a sufficiently long period of vibration. This wire supports a horizontal sheet of paper, which vibrates with the wire. If this paper be at rest while a torsional oscillation is given to the vertical wire under test, the jet of liquid will trace a circle on the paper. But if the paper now oscillates on the whole transversely to the motion of the jet, a waved curve will be traced, which crosses the circle at each semi-vibration. The interval of time between two successive crossings is constant (equal to the period of semi-vibration of the steel wire), and we can thus obtain a comparison of the times of outward and inward motions over a given range.

Two of these curves are shown in fig. 16. The part of a curve which corresponds to the outward motion can easily be distinguished from that which corresponds to the inward motion by its greater amplitude. In the first curve, 20 semi-vibrations take place in the range AB in the outward motion, while 20 take place in the range CA in the inward motion. The difference BC corresponds (allowing for the slight difference at the end A) to about one-third of a semi-vibration. Thus *the outward motion over the range* AC *occupies less time than the inward motion over the same range*, the difference being about 1 in 60.

Result of Heating the Wire to Redness.

[*Added* 18*th July* 1898.—It is to be expected that the molecular freedom which is introduced by heating the wire to redness will undo, to a great extent at least, the effect of fatigue. Before testing this point the wire was subjected to greater fatigue than on any previous occasion, and an experiment was then made on the date 1.7.98. The results were

$$a=4, \quad n=1\cdot015, \quad b=89\cdot6, \quad nb=91, \quad y_\circ=36\cdot7.$$

Thus by excessive fatigue the value of b was made smaller than it had ever been, while n, as formerly under such conditions, approximated to unity.

On the date 14.7.98 the wire was heated to redness by a Bunsen flame, the lead ring being removed to prevent stretching. An experiment was then made, and the results were

$$a=7, \quad n=1\cdot253, \quad b=680, \quad nb=852, \quad y_\circ=43\cdot4.$$

A comparison with the results given in the last column of Table IV. shows that b has become much more than twice as large as the greatest previous value.

It is interesting to compare this result with the results of two experiments made on the date 19.7.93, but not published in the first paper. In these experiments the wire hung inside a long solenoid composed of two similar coils of stout copper wire. In the first experiment a heavy current was run, in opposite directions, through the coils. The effect was to maintain the wire at a temperature of about 80° C. The results were

$$a=2, \quad n=1\cdot747, \quad b=536, \quad nb=936.$$

The difference between the conditions now considered and those above described is that now the wire is *maintained* at a comparatively high temperature during the experiment, while formerly it was heated to redness and was then experimented upon *when cold*. Though b is not quite so large in the latter case as in the former, n is considerably greater than formerly—so much so that nb is greater in the case now under discussion than in the other. Hence, when the temperature is maintained high, the loss of energy

per oscillation is much greater at large angles, much less at small angles, than it is when the temperature is normal, even after heating to redness.

In the second of the two experiments, performed immediately after the first, the only change made was that the current was sent in the *same* direction round the two coils. Thus, in addition to the maintenance of the wire at a temperature of about 80° C., *a steady state of magnetisation was maintained.* The results were

$$a=2, \quad n=2{\cdot}312, \quad b=2210, \quad nb=5110.$$

The effects just described are, therefore, in all respects greatly intensified. The molecular theory of magnetisation would lead one to expect decreased loss of energy at small angles, and increased loss at high angles, when the magnetisation is great.]

Theory of the Oscillations of an Imperfectly-Elastic Solid.

The first attempt at a theoretical investigation of the properties of a ductile solid was made by James Thomson (*Camb. and Dub. Math. Journ.*, 1848) in a paper "On the Strength of Materials, as influenced by the existence or non-existence of certain Mutual Strains among the Particles composing them." In applying his investigation to the case of torsion of a wire, he assumed that a certain definite tangential stress per unit area could be sustained without the production of permanent distortion, while an infinitesimal increase of the stress over this value caused continuous sliding until the stress diminished to the given definite value. In this way he explained the existence of elastic limits, and the greater strength of a wire as regards torsion in one direction or the opposite.

A mathematical development of Maxwell's views of the molecular constitution of a material substance is given by J. G. Butcher (*Proc. Lond. Math. Soc.*, vol. viii.) in a paper "On Viscous Fluids in Motion." In it, molecular groups are considered as consisting of two classes—those in which finite strain can be sustained without rupture, and those in which no strain can be sustained; and the properties of substances are regarded as depending upon the relative proportions in which those groups are present. The investigation deals only with those cases in which fluidity is manifest. The question of "elastic after-action" is included.

In the present investigation, the question of an imperfectly-elastic solid is alone considered, and elastic after-action is neglected. The case of torsion of a wire is explicitly developed. The fact that the period of oscillation had no effect on the experimental results obtained in the preceding part of the paper justifies the omission of the consideration of after-action in the application of the theory to these cases.

The time which elapses between the breaking down of a group and its formation into a new configuration is regarded as being zero in comparison with the time of motion of the wire through any finite range.

Consider unit length of the wire. Let ξ be the relative linear displacement per unit length at which a particular group breaks down, and let $\nu d\xi$ be the number of such groups which break in the increment of displacement $d\xi$. Then, in the element of volume $2\pi r dr$, the number $2\pi\nu r dr d\xi$ break down in the increment $d\xi$. Let θ be the angular distortion per unit length of the wire. Then $r\theta$ is the shear in the element of volume under consideration. Let

$$\xi = \frac{1}{m} r\theta, \quad \xi' = \frac{1}{m-1} r\theta,$$

where m is a whole number. If we assume that a group which breaks at the shear ξ is, on the average, formed again into a group which also breaks at the shear ξ, those groups which break at ξ and ξ' will also break at $r\theta$. Now take

$$\xi'' = \xi + p(\xi' - \xi) = \xi\left(1 + \frac{p}{m-1}\right),$$

where p is a proper fraction.

A group which breaks at ξ'', has had, when the total shear is $r\theta$, $m-1$ breaks, its last being at $(m-1)\xi'' = (m-1+p)\xi$. The shear to which it is subjected, when the total shear is $r\theta$, is therefore

$$(m-1)(\xi' - \xi'') = (1-p)\xi.$$

Hence, if we divide the shear $\xi' - \xi$ into an infinite number of equal parts $d\xi$, the average value of p is $\frac{1}{2}$, so that the average value of the stretch to which the group which breaks at ξ'' is subjected, when the total shear is $r\theta$, is $r\theta/2m$.

Now the number $2\pi\nu r dr d\xi$, when summed over the range corresponding to two consecutive values of m, becomes

$$\frac{2\pi\nu r dr}{m(m-1)} \cdot r\theta.$$

So, if the stress to which a group is subjected when it sustains a shear x is, on the average, kx, the total stress for the above number of groups is

$$\frac{\pi k\nu r^3\theta^2 dr}{m^2(m-1)}.$$

And the total stress due to groups which break at shears lying between 0 and $r\theta$ is

$$\theta^2\pi k\nu \sum_{2}^{\infty} \cdot \frac{1}{m^2(m-1)} \int_0^a r^3 dr = \frac{1}{4}\pi k\nu\theta^2 a^4 \sum_{2}^{\infty} \frac{1}{m^2(m-1)}, \quad . \quad . \quad . \quad . \quad (1)$$

where a is the radius of the wire, and ν and k are assumed to be constants.

If N be the total number of groups per unit volume, the number of unbroken groups is, in the volume $2\pi r dr$,

$$\left(\mathrm{N} - \int_0^{r\theta} \nu d\xi\right) 2\pi r dr;$$

and the total stress due to such groups is

$$\int_0^a (\mathrm{N} - \nu r\theta)\cdot kr\theta\cdot 2\pi r dr = 2\pi k a^3\left(\frac{\mathrm{N}}{3}a\theta - \frac{\nu}{4}a^2\theta^2\right). \qquad (2)$$

The total force tending to diminish the torsion is therefore

$$\frac{2}{3}\pi k\mathrm{N}a^2(a\theta) - \frac{1}{4}\pi k\nu a^2\left[2 - \sum_2^\infty \frac{1}{m^2(m-1)}\right](a\theta)^2.$$

The *single* force which, acting at the distance a from the axis, would equilibrate this is

$$\frac{1}{2}\pi k\mathrm{N}a^2(a\theta) - \frac{1}{5}\pi k\nu a^2\left[2 - \sum_2^\infty \frac{1}{m^2(m-1)}\right](a\theta)^2.$$

$$= \frac{1}{2}\pi k\mathrm{N}a^2(a\theta) - \frac{1}{5}\pi k\nu a^2\sum_1^\infty \frac{1}{m^3}(a\theta)^2. \qquad (3)$$

Hence *the deviation from Hooke's Law is represented by a negative term involving the square of the distortion*, provided that the quantity ν is constant.

But ν is the rate at which groups break down per unit change of distortion. Thus (3) gives the theoretical deviation from Hooke's Law *when the range of distortion at which a group breaks down is, on the average for all groups, uniformly distributed over all possible ranges.*

If ν were zero there would be no internal loss of energy in the wire; and, if the wire were once set in oscillation, the oscillations would, so far as this cause is concerned, continue for ever without any loss of amplitude. If ν is very small, the difference between the quantities of energy stored up in the wire in two successive maximum twists is practically proportional to ydy/dx, where y is the scale-reading and x represents number of oscillations, since Hooke's Law is nearly obeyed; and we can easily prove (see below) that the loss of energy in an outward oscillation is proportional to the cube of the distortion. Also, since, by our fundamental assumptions, every group which broke down at a certain stage in the outward motion breaks down again at the same point in the inward motion, the total loss of energy, in the form of heat, in the inward motion to the zero is equal to that in the outward motion from zero. Hence we get $-bdy = y^2dx$, which gives

$$y(x+a) = b.$$

This is, as we have seen, precisely the equation which was found experimentally to connect range of oscillation with number of oscillations when the wire is greatly fatigued. If, therefore, our theoretical assumptions correctly represent the physical conditions, *the effect of great fatigue is to produce averagely uniform distribution of breaking range over all possible values.*

The apparatus which was used in the experimental investigations was not suitable for the purpose of testing the expression (3) directly in its application to the torsion of wires. Table VII. has been drawn up for me by Mr P. S. HARDIE, formerly Neil Arnott scholar in the Physical Laboratory, to test the applicability to the bending of bars of the equation

$$y = ax - bx^2,$$

where y represents distorting force and x represents distortion. The data used in the calculation are some of those given by HODGKINSON and FAIRBAIRN in the *B. A. Reports*, 1837. The columns headed x and y give observed values of these quantities; the columns headed y' give calculated values of y. The correspondence is extremely close, in some cases remarkably so, when it is considered that any flaw in the homogeneity of the material tends to introduce irregularities in the action under stress. Fig. 15 exhibits graphically the results in one case. The full curve represents a curve $y = ax - bx^2$, and the points on or near it are obtained from the experiments. The straight full line in the diagram represents the Hooke's Law line $y = ax$. The coordinate, $y = a^2/4b$, of the vertex of the parabola corresponds theoretically to the breaking stress. The material always, as is to be expected, breaks at a smaller stress.

We have now to investigate the inward motion. At any stage, all groups which give rise to an inward force in the outward motion give rise to the same inward force in the inward motion, provided that their last breaking-point has not been repassed. On the other hand, those groups whose last breaking-point has been repassed do not exert an inward force, but in general exert an outward force. Hence the inward force at any stage on the inward motion to zero is less than the inward force at the same stage on the outward motion. Thus we deduce at once from the theory the observed result that *the time of outward motion over a given range is less than the time of inward motion over the same range.*

Let us suppose now that the angular distortion ϕ, in the inward motion, has become less than half the maximum angular distortion θ. *Every* group which broke down in the outward motion is now exerting an outward force. In the volume $2\pi r dr$, since we are assuming that the breaking range of distortion for different groups is, on the average, uniformly distributed over all possible values, all groups which broke *first* between ϕ and θ are now exerting on the average an outward force $\frac{1}{2}kr(\theta - \phi)$. All those which broke at a range less than ϕ are now exerting an outward force which is proportional to the distance between $r\phi$ and their *last* breaking-point on the inward motion. To find the total value of this force, consider $m\xi = r\phi, (m-1)\xi' = r\phi$. A group which broke at

$$\xi'' = \xi + p(\xi' - \xi) = \frac{r\phi}{m}\left(1 + \frac{p}{m-1}\right)$$

had its nearest breaking-point outside $r\phi$ at $m\xi''$. Its distortion is therefore $m\xi'' - r\phi = pr\phi/(m-1)$. Now, at the fixed point $r\phi$, when ξ'' ranges over $\xi - \xi'$, p takes all values from 0 to 1 uniformly, so that its average value is $\frac{1}{2}$. Hence we find that the outward

pull exerted by all groups which broke first in the range $\xi' - \xi$ is

$$\int_0^a kv(\xi' - \xi)\frac{1}{2}\frac{r\phi}{m-1}\,.\,2\pi r dr = \frac{1}{4}\pi kva^2(a^2\phi^2)\frac{1}{m(m-1)^2}.$$

Thus the total outward force due to those groups whose breaking-range ξ is less than $r\phi$ is

$$\frac{1}{4}\pi kva^2(a^2\phi^2)\sum_2^\infty \frac{1}{m(m-1)^2} = \frac{1}{4}\pi kva^2(a^2\phi^2)\sum_2^\infty.\frac{1}{m^2}.$$

The single force, equivalent to this, acting at a distance a from the axis, is

$$\frac{1}{5}\pi kva^2(a^2\phi^2)\sum_2^\infty.\frac{1}{m^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The outward force due to groups which broke first between θ and ϕ is

$$\int_0^a \frac{1}{2}kr(\theta - \phi)\,.\,2\pi r dr\,.\,vr(\theta - \phi) = \frac{1}{4}\pi kva^2[a^2(\theta - \phi)^2]\,.$$

Referred to a this becomes

$$\frac{1}{5}\pi kva^2[a^2(\theta - \phi)^2]\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The whole inward force due to unbroken groups is

$$\int_0^a (\mathrm{N} - vr\theta)\,.\,2\pi r dr\,.\,kr\phi = 2\pi ka^2(a\phi)\left[\frac{1}{3}\mathrm{N} - \frac{1}{4}v(a\theta)\right].$$

When referred to distance a this becomes

$$2\pi ka^2(a\phi)\left[\frac{1}{4}\mathrm{N} - \frac{1}{5}v(a\theta)\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The total inward force is therefore

$$\frac{1}{2}\pi k\mathrm{N}a^2(a\phi) - \frac{1}{5}\pi kva^2(a^2\phi^2)\,.\,\sum_1^\infty\frac{1}{m^2} - \frac{1}{5}\pi kva^2(a^2\theta^2)\,. \quad . \quad . \quad . \quad . \quad (7)$$

By comparison of the expressions (3) and (7) we see that *when, in the inward motion, the range is less than half its maximum value, the inward force is less than the inward force at the same stage on the outward motion by an amount which depends only on the square of the maximum range.*

When, in the inward motion, the zero is reached, every group which has broken breaks and re-forms into its initial condition, so that the oscillation proceeds, as formerly, on the other side of the zero, but with less initial energy,—so giving rise to the lessening of amplitude.

Now, as a given increase of maximum range decreases the inward force at any stage of the inward motion more and more as that range is greater, the time of inward motion increases when the range increases. But the form of (3) shows that the time of outward motion is less when the range of oscillation is small than when it is large. Therefore *the period of complete oscillation is greater for large oscillations than for small.* This was shown in the first paper. KUPFFER pointed it out first in 1853.

The result that the zero of oscillation is a point at which groups re-form into their original condition explains the fact of the constancy of that zero which was found to obtain as oscillations proceed (see Second Paper).

The expression (7) vanishes when

$$\left(a\phi - \frac{5N}{4\nu\sum_1^\infty \frac{1}{m^2}}\right)^2 = \left(\frac{5N}{4\nu\sum_1^\infty \frac{1}{m^2}}\right)^2 - \frac{a^2\theta^2}{\sum_1^\infty \frac{1}{m^3}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

This is, according to the theory, the relation which connects the angle of set with the angle of maximum twist, provided that the former does not exceed half the latter, and provided also that ν is constant—a condition which seems to hold, as we have seen, when the wire is greatly fatigued. This equation represents an ellipse whose semi-axes have a ratio of about 13 to 10, and would imply that the wire would flow round under the action of continued stress when the set equalled about ten-thirteenths of the distortion, if we could apply the equation to sets beyond half distortion (see Note).

If the inward motion were stopped just short of the zero, and the wire were then given an outward motion, the conditions differ from those in the first outward motion. When the angle reaches a value ψ, equation (6) gives the inward force due to unbroken groups if ϕ be replaced by ψ. With the same substitution, (5) represents the outward pull due to groups which broke first between ψ and θ. So also, ψ being substituted for θ, (1) gives the inward pull due to groups which broke between 0 and ψ. Hence, the expression in (1) being referred also to distance a from the axis, the total inward force in this case is

$$\tfrac{1}{2}\pi k N a^2 (a\psi) - \tfrac{1}{3}\pi k\nu a^2 (a^2\psi^2) \sum_2^\infty \frac{1}{m^2} - \tfrac{1}{3}\pi k\nu a^2 (a^2\theta^2) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

This differs from the expression (7) in the multiplier of the middle term. The value of $\sum_1^\infty \frac{1}{m^2}$ is very closely 5/3 and that of $\sum_2^\infty \frac{1}{m^2}$ is closely 2/3.

The expressions (3) and (9) have identical values when $\psi = \theta$, after which, the angle θ not being exceeded, the inward motion again obeys the law of force given

by (7); the next outward motion, the in motion being stopped just short of the zero, again obeys the law of force given by (9); and so on. By taking $\sum\limits_{2}^{\infty}.\frac{1}{m^2}$ instead of $\sum\limits_{1}^{\infty}.\frac{1}{m^2}$ in equation (8) we get an expression for the angle of set in the first part of the outward motion under these circumstances.

We can easily get a simple graphical construction for the two extreme positions of set. Plot forces as abscissæ and angles as ordinates. Draw the Hooke's Law line as indicated by the first term of (3). Draw also the parabolic curve given by (3), and the parabolic curve indicated by the first two terms of (9). Take three-fifths of the difference of abscissæ of the Hooke's Law line and the former parabola at the ordinate corresponding to the maximum angle θ, and plot it along the line of abscissæ. The ordinate drawn through the point so found intersects the two parabolæ at points whose ordinates are the extreme angles of set. The method is shown in fig. 15.

The dotted curve in fig. 15 is the second parabola above referred to, the full curve being the first. The position of set being taken as origin, the dotted curve does not greatly differ from a straight line, the deviations at the larger forces being in the direction of too great distortion. This result explains WIEDEMANN's observation (*Philosophical Magazine*, vol. ix., 1880) that, *after a wire has been twisted a few times in opposite directions alternately by a given couple, and is then twisted by increasing couples in the direction of the last twist, Hooke's Law is nearly obeyed, provided the original couple is not exceeded, the slight deviations being in the direction of too great twist.*

In order to deduce the expression

$$y^n(x+a)=b$$

as the more general relation connecting range of oscillation with number of oscillations, we have only to assume that the quantity ν, employed in the preceding investigation, varies as a power of the strain. Take $\xi = r\theta/(m+p)$ where m is a whole number and p is a proper fraction; and, instead of ν, let us write

$$\nu\left(\frac{r\theta}{m+p}\right)^{\mu}$$

where ν and μ are regarded as constants. Each group which breaks at ξ has, when it breaks, potential energy $\frac{1}{2}k\xi^2$, which is transformed into heat. Also each such group, p varying from 0 to 1, breaks m times. Hence the heat developed in the range 0 to θ, is, in the volume $2\pi r dr$,

$$\tfrac{1}{2}km\int_{p=1}^{p=0}\left(\frac{r\theta}{m+p}\right)^2.\nu\left(\frac{r\theta}{m+p}\right)^{\mu}d\left(\frac{r\theta}{m+p}\right).2\pi r dr = \pi k\nu\frac{(r\theta)^{3+\mu}}{3+\mu}\,\frac{r dr}{m^{2+\mu}(m+1)^{3+\mu}}\,.$$

The total loss of energy is therefore

$$\frac{\pi k \nu a^2 (n\theta)^{3+\mu}}{(3+\mu)(5+\mu)} \sum_1^{\infty} \frac{m}{[m(m+1)]^{3+\mu}}.$$

If this loss is a small fraction of the whole energy we may write it proportional to $\theta d\theta/dx$, and, by integration, obtain, in the former notation, the result

$$y^{3+\mu}(x+a) = b.$$

The theory therefore indicates that *n is greater or less than unity, according as groups breaking at large distortions are more or less numerous than groups breaking at small distortions.*

We can easily, as above, determine the more general relation which connects set with torsion, but it is sufficient to note that the preceding considerations justify, from the point of view of theory, the adoption of the approximate expression used in the first paper on this subject, and that they are therefore justified, in turn, by the experimental confirmation therein given.

It is not to be supposed that the agreement of the results of the above theory with the results of observation necessarily proves the truth of the particular assumptions therein made. The object of the investigation is rather to show how well a theory based upon simple and reasonable assumptions concerning molecular statistics can account for general phenomena exhibited by imperfectly elastic solid media.

NOTE. *Added* 6*th October* 1898.

It is of interest to determine the general law of motion at all stages of the inward motion. Let θ and ϕ have the same meanings as formerly, and take

$$r\theta = (1+p)r\phi$$

with the condition

$$\frac{1}{p} = \mu + \lambda,$$

where μ is a whole number and λ is a proper fraction. Consider the various stages $r\phi/(m+1)$ to $r\phi/m$, where m is a whole number.

A group which breaks at

$$\frac{r\phi}{m+1} + x\frac{r\phi}{m(m+1)}$$

has its $(m+1)^{\text{th}}$ break at

$$r\phi + x\frac{r\phi}{m}.$$

For all values of x from 0 to 1 this point lies between $r\phi$ and $r\theta$, provided that we have

$$m > \frac{1}{p}.$$

When the stage ϕ on the inward motion is reached, all such groups exert outward force, and their average stretch is

$$\frac{1}{2}\left[\frac{m+1}{m} r\phi - r\phi\right] = \frac{1}{2}\frac{r\phi}{m}.$$

The total outward pull due to them is therefore

$$\sum_{n+1}^{\infty} \cdot \int_0^a 2\pi r dr \cdot \frac{1}{2}k\frac{r\phi}{m} \cdot \nu \frac{r\phi}{m(m+1)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

the summation being with respect to m.

When we have

$$m < \frac{1}{p},$$

we must take the fraction x so that its largest value is given by $r\phi + xr\phi/m = r\theta$, *i.e.*,

$$x = mp.$$

Then the number of groups

$$\nu mp \frac{r\phi}{m(m+1)} = \nu \frac{r\theta - r\phi}{m+1}$$

break in the range $r\phi$ to $r\theta$ with an average stretch $\frac{1}{2}(r\theta - r\phi)$. Hence their outward pull is

$$\sum_1^{n} \frac{1}{m+1} \cdot \int_0^a 2\pi r dr \cdot \frac{\nu}{2} k (r\theta - r\phi)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In the case of the remaining number

$$(1 - mp)\nu \frac{r\phi}{m(m+1)} = \left[\frac{r\phi}{m} - \frac{r\theta}{m+1}\right]\nu,$$

we have to consider the m^{th} break. Now the m^{th} break of a group which broke at $r\phi/(m+1) + mpr\phi/m(m+1)$ occurs at $mr\theta/(m+1)$, so that the average stretch for this number is

$$\frac{1}{2}m\left[\frac{r\phi}{m} - \frac{r\theta}{m+1}\right].$$

Hence the total inward pull of these groups is

$$\sum_1^{n} \cdot \int_0^a 2\pi r dr \cdot \nu \frac{1}{2} k m \left[\frac{r\phi}{m} - \frac{r\theta}{m+1}\right]^2 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

To these expressions we have to add the outward pull of groups which break only between $r\phi$ and $r\theta$. This is

$$\int_0^a 2\pi r dr \,.\, \frac{1}{2} k\nu (r\theta - r\phi)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

By integration of the expressions (10), (11), (12), and (13), and by supposing, as formerly, that the forces act at a distance a from the axis, we find that the total inward force is

$$\frac{1}{5}\pi k\nu a^4 \left\{ \sum_1^\mu m \left[\frac{\phi}{m} - \frac{\theta}{m+1}\right]^2 - (\theta - \phi)^2 \sum_1^\mu \frac{1}{m+1} - \phi^2 \sum_{\mu+1}^\infty \frac{1}{m^2(m+1)} - (\theta-\phi)^2 \right\} + 2\pi k a^3 \phi \left[\frac{N}{4} - \frac{1}{5}\nu a \theta\right]$$

if we take account of the pull (6) due to unbroken groups. This can be put in the form

$$\frac{1}{2}\pi k N a^2 (a\phi) - \frac{1}{5}\pi k\nu a^2 (a^2\phi^2) \sum_{\mu+1}^\infty \frac{1}{m^2} - \frac{1}{5}\pi k\nu a^2 (a^2\theta^2) \sum_1^{\mu+1} \frac{1}{m^2} \quad . \quad . \quad . \quad (14)$$

which reduces, when we put $\mu = 0$, to the expression (7) applying to the second half of the inward motion.

The points $\left(1 - \frac{1}{m}\right) r\theta$ are points such that, in the intermediate ranges, the multipliers of the second and third terms in (14) remain constant. The sudden changes in the magnitudes of these terms are equal and opposite. For, when ϕ reaches the value $\mu\theta/(\mu + 1)$, λ having become zero in the expression $1 + \mu + \lambda$, μ is to be suddenly changed to $\mu - 1$ in the affixes of the summations, so that the second term is suddenly increased by the amount

$$\frac{1}{5}\pi k\nu a^2 \left(a^2 \frac{\mu^2}{(\mu+1)^2} \theta^2\right) \frac{1}{\mu^2} = \frac{1}{5}\pi k\nu a^2 (a^2\theta^2) \frac{1}{(\mu+1)^2}$$

which is also the decrease of the third term. Thus the force varies continuously.

The amount by which (14) differs from (3) at any definite value of the angle is

$$-\frac{1}{5}\pi k\nu a^2 \left[\sum_1^{\mu+1} \frac{1}{m^2} \,.\, a\theta^2 - \sum_1^\mu \frac{1}{m^2} \,.\, a^2\phi^2\right].$$

This is therefore the continuously varying expression for the defect of the inward force at a given stage in the inward motion from the inward force at the same stage in the outward motion.

The limiting boundary of the space included by the series of ellipses represented by equating (14) to zero indicates the general relation between torsion and set when ν is constant. These ellipses intersect consecutively at points where $2\phi = \theta$, $3\phi = 2\theta$, $4\phi = 3\theta$, etc. At these points the rate of variation of set with torsion changes suddenly.

TABLE I.—*Results of the First Series of Experiments.*

Date.	a	n	b	nb	y_0
8.6.96	5	1·148	178	204	44·5
16.7.96	4	1·037	100	104	35·1
18.7.96	4	1·030	99·7	103	32·5
20.7.96	4	1·040	99	103	31·5
21.7.96	4	1·045	100	104	30·1
22.7.96	13	1·033	93·5	96·6	8·05 ?
23.7.96	7	1·040	101	105	16·5
23.7.96	7	1·020	98	100	16·5
24.7.96	3	1·067	99	106	45·2
27.7.96	5	1·000	97	97	29·0
27.7.96	4	1·060	100	106	29·0
28.7.96	5	1·017	100	102	26·2
30.7.96	6	1·033	100	103	20·3

TABLE II.—*Test of the Results of the First Series.*

16.7.96			18.7.96			20.7.96		
17·9	19·8	18·3	18·2	18·5	17·6	17·6	18·4	17·3
15·1	15·5	15·3	15·3	14·9	14·9	14·8	14·9	15·0
13·0	13·0	13·2	13·2	12·9	12·9	12·8	12·8	12·7
10·2	10·2	10·3	10·3	10·2	10·1	10·0	10·1	10·0
8·4	8·4	8·5	8·5	8·4	8·4	8·3	8·4	8·3
6·7	6·6	6·7	6·7	6·6	6·7	6·6	6·6	6·6
5·0	4·9	5·0	5·0	4·9	5·0	4·9	4·9	4·9
4·0	4·0	4·0	4·0	4·0	4·0	3·9	3·9	3·9
3·3	3·3	3·3	3·3	3·3	3·3	3·2	3·2	3·3
2·8	2·9	2·9	2·8	2·8	2·8	2·8	2·9	2·8
2·5	2·5	2·5	2·5	2·5	2·5	2·5	2·6	2·5

TABLE II.—*Continued.*

21.7.96			22.7.96			23.7.96		
17·5	18·4	17	6·3	6·4	6·2	11·4	11·8	11·6
14·7	14·8	14·4	5·9	5·9	5·9	10·2	10·4	10·4
12·7	12·8	12·5	5·5	5·6	5·5	9·2	9·4	9·4
10·0	10·0	9·9	5·0	4·9	5·0	7·7	7·7	7·8
8·3	8·3	8·2	4·6	4·4	4·5	6·7	6·6	6·7
6·6	6·4	6·6	3·9	3·9	4·0	5·5	5·4	5·6
4·9	4·9	4·9	3·2	3·2	3·3	4·3	4·3	4·3
3·9	3·9	3·9	2·7	2·7	2·8	3·6	3·6	3·6
3·3	3·3	3·3	2·4	2·4	2·5	3·0	3·0	3·0
2·8	2·9	2·8	2·1	2·2	2·2	2·6	2·7	2·6
2·5	2·6	2·5	1·9	2·0	2·0	2·3	2·4	2·3
2·2	2·3	2·2	1·7	1·9	1·8	2·1	2·1	2·1
2·0	2·0	2·0	1·6	1·8	1·7	1·9	1·9	1·9
1·8	1·8	1·9	...	...	...	...	...	...

23.7.96			24.7.96			27.7.96		
11·7	11·8	11·6	17·7	20·3	19·1	16·7	16·8	16·7
10·4	10·4	10·4	14·8	15·5	15·9	14·3	14·1	14·2
9·4	9·4	9·4	13·1	13·3	13·6	12·5	12·3	12·4
7·8	7·7	7·8	10·3	10·3	10·6	10·0	9·7	9·8
6·7	6·6	6·7	8·4	8·5	8·7	8·3	8·1	8·2
5·6	5·4	5·6	6·6	6·5	6·8	6·7	6·4	6·5
4·2	4·3	4·3	4·9	4·7	5·1	5·0	4·8	4·9
3·6	3·6	3·6	3·9	3·9	4·0	4·0	3·8	3·9
3·0	3·0	3·0	3·2	3·2	3·3	3·3	3·2	3·3
2·6	2·7	2·6	2·8	2·8	2·9	2·9	2·8	2·8
2·3	2·4	2·3	2·5	2·5	2·5	2·5	2·5	2·5
2·0	2·1	2·1	2·2	2·2	2·2	2·2	2·2	2·2
1·9	1·9	1·9	1·9	1·9	2·0	2·0	2·0	2·0
			1·8	1·8	1·8	1·8	1·8	1·8

TABLE II.—*Continued.*

27.7.96			28.7.96			30.7.96		
16·7	16·8	16·7	15·9	16·7	15·1	13·1	13·4	13·4
14·1	14·1	14·2	13·1	13·7	13·0	11·5	11·7	11·8
12·2	12·3	12·4	12·0	11·9	11·5	10·3	10·4	10·5
9·6	9·7	9·8	9·8	9·5	9·3	8·5	8·5	8·6
8·0	8·1	8·2	7·5	8·2	7·8	7·2	7·1	7·3
6·4	6·4	6·5	6·4	6·3	6·3	5·9	5·8	6·0
4·8	4·8	4·9	4·9	4·7	4·7	4·5	4·4	4·6
3·8	3·8	3·9	4·0	3·8	3·8	3·7	3·7	3·7
3·2	3·2	3·3	3·2	3·1	3·2	3·1	3·1	3·1
2·8	2·8	2·8	2·8	2·8	2·8	2·7	2·7	2·7
2·4	2·5	2·5	2·5	2·5	2·4	2·4	2·4	2·4
2·2	2·2	2·2	2·2	2·2	2·3	2·1	2·0	2·2
2·0	2·0	2·0	2·0	2·0	2·0	1·9	1·9	2·0
1·8	1·8	1·8	...	...	...	1·7	1·7	1·8

TABLE III.—*Tests of Initial Deviations from Formulæ.*

Date.	y_0	N	p	a	Date.	y_0	p	a
16.7.95	37·1	1	2·13	6	9.12.95	37·2	0·35	3
17.7.95	51·3	10	2·06	4	12.12.95	36·8	1·25	3
18.7.95	44·4	20	2·00	4	17.12.95	14·2	0·00	9
19.7.95	41·2	30	2·49	5	18.12.95	14·3	−0·70	9
20.7.95	36·6	1	2·09	5	19.12.95	9·6	0·20	22
20.7.95	48·7	50	1·58	3	19.12.95	7·0	−0·20	25
20.7.95	39·7	1	1·47	4	20.12.95	5·3	0·80	80
22.7.95	40·0	80	1·29	3	20.12.95	3·0	0·00	120
23.7.95	42·0	120	0·73	2	24.12.95	1·6	−2·00	219
25.7.95	30·2	160	0·23	2	24.12.95	8·5	−0·30	47
26.7.95	38·7	1	1·98	4	...	...	...	...
26.7.95	43·9	200	0·93	2	...	...	...	...
27.7.95	41·5	50	0·95	2	...	...	...	...

TABLE IV.—*Re-calculated Data for Table I. in the First Paper.*

Date.	n'	a'	b'	b''	n	a	b
5.7.93	1·05	6·5	574	543	1·02	7·5	196
7.7.93	1·18	7·5	802	723	1·13	8·5	231
10.7.93	1·18	6·6	802	770	1·16	6·6	238
10.7.93	1·18	6·4	802	847	1·20	6·4	246
10.7.93	1·18	6·4	802	820	1·19	6·4	247
14.7.93	1·18	7·3	842	822	1·17	7·3	252
14.7.93	1·18	6·7	802	781	1·167	6·7	240
17.7.93	1·32	4·4	1074	1080	1·326	4·4	283
18.7.93	1·18	6·6	802	820	1·19	6·6	247
18.7.93	1·18	7·0	802	824	1·19	7·0	248
18.7.93	1·40	2·6	761	738	1·38	3·0	183

TABLE V.—*Data for Second Series of Experiments.*

Date.	a	n	b	nb	y_0
14.10.97	7	0·9	129	116	37·5
15.10.97	7	0·89	117	104	39·0
18.10.97	7	0·87	107	93	40·3
19.10.97	6	0·95	119	113	33·9
20.10.97	6	0·92	112	103	43·6
21.10.97	6	0·935	117	109	39·2
22.10.97	6	0·917	110	101	41·2
25.10.97 (1)	6	0·95	122	116	39·0
25.10.97 (2)	6	0·91	107	97	41·0
26.10.97	6	0·92	111	102	39·3
27.10.97 (1)	6	0·912	107	98	40·1
27.10.97 (2)	6	0·92	111	102	43·7
28.10.97	5	0·96	105	101	37·1 (N = 5)

TABLE V.—*Continued.*

Date.	a	n	b	nb	y_0
29.10.97 (1)	5	0·957	104	100	38·7 (N = 5)
29.10.97 (2)	5	0·957	101	96	38·3 (N = 10)
1.11.97 (1)	5	0·985	113	111	39·6
1.11.97 (2)	6	0·990	119	118	22·9
2.11.97 (1)	7	0·970	127	123	28·4
2.11.97 (2)	5	0·975	105	102	37·1
3.11.97 (1)	5	1·000	114	114	40·5
3.11.97 (2)	10	0·990	124	123	12·8
4.11.97 (1)	7	0·965	119	115	20·7
4.11.97 (2)	5	0·968	100	97	35·7 (N = 20)
5.11.97	4	1·022	100	102	37·1 (N = 40)
8.11.97 (1)	5	1·025	116	119	40·7
8.11.97 (2)	5	0·985	99	98	36·1 (N = 20)
9.11.97	12	0·992	148	147	12·7
10.11.97 (1)	12	1·010	165	167	13·1
10.11.97 (2)	17	0·913	152	139	11·0
10.11.97 (3)	16	0·900	147	132	11·2
11.11.97 (1)	5	1·012	116	117	35·6
11.11.97 (2)	15	0·950	125	119	9·3
12.11.97 (1)	11	1·008	101	102	9·4 (N = 40)
12.11.97 (2)	4	1·020	99	101	39·0
15.11.97 (1)	50	0·680	213	145	8·6
15.11.97 (2)	4	1·042	127	132	39·8
16.11.97 (1)	17	1·017	166	175	9·8
16.11.97 (2)	4	1·030	101	104	36·5 (N = 60)
17.11.97 (1)	11	0·953	134	128	14·3
17.11.97 (2)	10	0·982	136	134	12·2
17.11.97 (3)	4	1·030	106	109	32·2
18.11.97 (1)	30	0·857	151	129	9·8
18.11.97 (2)	18	0·950	159	152	10·0
19.11.97	220	0·270	562	152	4·3
22.11.97 (1)	60	0·523	313	164	9·6
22.11.97 (2)	25	0·695	164	114	15·2
23.11.97	20	0·740	160	118	17·0
24.11.97	8	0·925	135	125	29·9
25.11.97	6	0·968	123	119	38·0
14.12.97	300	0·590	655	386	3·5
15.12.97	6	1·010	144	145	34·3
9.2.98	220	0·363	600	218	4·5

TABLE VI.—*Former and Re-calculated Data for Table II. in the First Paper.*

Date.	65	60	55	50	45	40	35	30	25	20	17	15	12	10	9	8	7	6	5	4
5.7.93		7·7	8·5	9·3	10·5	11·7	13·7	16·0	19·5	24·7	28·5	33·2	42·1	51·0	57·1	64·5	...	...	...	...
		8·0	8·5	9·2	10·2	11·5	13·3	16·0	19·6	24·6	28·9	33·3	42·0	51·6	57·6	65·6	...	...	...	...
		7·3	9·0	9·4	10·2	11·6	13·4	15·9	19·4	24·6	29·2	33·3	42·1	50·9	56·7	64·1	...	...	...	...
7.7.93				7·9	8·9	10·3	12·1	14·5	18·0	23·4	27·6	32·9	42·7	53·0	60·0	69·0	...	...	...	...
				8·0	8·9	10·3	12·1	14·9	18·2	23·4	28·1	32·4	42·5	53·0	60·0	69·5	...	...	...	...
				7·7	8·8	10·2	12·0	14·5	18·0	23·5	28·4	32·9	42·6	52·6	59·4	68·0	...	...	...	...
10.7.93			7·1	7·9	8·9	10·3	12·1	14·5	18·0	23·4	27·6	32·9	42·7	53·0	60·0	69·0	...	...	...	...
			7·4	8·0	8·9	10·3	12·1	14·8	18·3	23·3	28·0	32·1	41·6	52·6	60·4	69·0	...	...	...	...
			7·4	8·2	9·3	10·7	12·5	14·9	18·4	23·9	28·8	33·3	43·1	53·3	60·7	69·9	...	...	...	...
10.7.93			7·1	7·9	8·9	10·3	12·1	14·5	18·0	23·4	27·6	32·9	42·7	53·0	60·0	69·0	...	...	...	...
			7·0	7·8	8·7	10·0	11·9	14·4	18·1	23·4	28·0	32·7	42·5	53·4	60·4	70·4	...	...	...	...
			6·9	7·7	8·8	10·1	11·9	14·3	17·8	23·3	28·3	32·9	42·9	53·4	60·7	69·9	...	...	...	...
10.7.93			7·1	7·9	8·9	10·3	12·1	14·5	18·0	23·4	27·6	32·9	42·7	53·0	60·0	69·0	...	...	...	...
			7·0	7·8	8·7	10·0	11·8	14·5	18·0	23·3	27·8	32·1	42·2	52·4	60·4	70·4	...	...	...	...
			7·0	7·8	8·8	10·2	11·9	14·3	17·8	23·2	28·2	32·7	42·6	52·9	61·4	70·6	...	...	...	...
14.7.93			7·4	8·3	9·3	10·8	12·7	15·2	18·9	24·5	29·0	34·5	44·8	55·6	63·0	72·4	...	...	...	...
			8·0	8·6	9·5	10·8	12·7	15·4	19·1	24·6	29·4	34·3	44·7	55·8	62·8	73·3	...	...	...	...
			7·6	8·5	9·6	11·0	12·8	15·4	19·0	24·7	29·9	34·5	44·9	55·5	62·9	72·2	...	...	...	...
14.7.93			7·1	7·9	8·9	10·3	12·1	14·5	18·0	23·4	27·6	32·9	42·7	53·0	60·0	69·0	...	...	...	...
			7·4	8·0	8·8	10·2	12·1	14·7	18·4	23·6	28·7	33·2	43·1	53·3	60·0	69·7	...	...	...	...
			7·3	8·1	9·2	10·5	12·3	14·8	18·2	23·7	28·6	33·1	43·0	53·2	60·1	69·0	...	...	...	...
17.7.93	4·35	4·8	5·3	6·1	7·1	8·2	9·85	12·0	15·3	20·5	24·8	30·1	40·4	51·4	59·0	69·0	82·3	...	...	...
	4·60	4·8	5·2	5·9	6·9	8·1	9·80	12·0	15·1	20·5	25·0	29·6	39·7	51·0	58·4	69·4	81·4	...	...	...
	4·3	4·7	5·3	6·0	6·9	8·1	9·7	11·9	15·1	20·3	25·2	29·8	40·0	51·0	58·7	68·5	81·8	...	...	...
18.7.93			7·1	7·9	8·9	10·3	12·1	14·5	18·0	23·4	27·6	32·9	42·7	53·0	60·0	69·0	...	...	...	...
			7·3	7·9	8·8	10·0	11·9	14·5	18·0	23·6	27·2	32·3	42·6	52·9	60·1	69·1	...	...	...	...
			7·0	7·8	8·8	10·2	11·9	14·3	17·8	23·2	28·2	32·7	42·6	52·9	61·4	70·6	...	...	...	...
18.7.93			7·1	7·9	8·9	10·3	12·1	14·5	18·0	23·4	27·6	32·9	42·7	53·0	60·0	69·0	...	...	...	...
			7·4	7·9	8·7	10·2	11·7	14·4	18·0	23·4	28·3	32·5	42·5	53·0	60·0	70·0	...	...	...	...
			7·0	7·8	8·9	10·2	12·0	14·4	17·9	23·3	28·3	32·8	42·8	53·2	61·7	71·0	...	...	...	...
18.7.93			2·8	3·2	3·7	4·3	5·2	6·4	8·4	11·7	14·0	17·2	23·4	30·3	35·1	41·4	49·9	61·9	79·9	109·3
			2·7	3·1	3·5	4·1	5·2	6·6	8·8	11·6	14·2	17·1	23·6	30·0	33·6	40·6	49·6	61·6	80·6	109·0
			2·5	2·9	3·5	4·1	5·1	6·4	8·3	11·4	14·4	17·2	23·5	30·4	35·2	41·5	49·9	61·9	79·9	108·6

TABLE VII.—*Results for Hodgkinson's and Fairbairn's Experiments.*

H. Exp. I, 1.

x	y	y'
3·7	3·2	3·17
5·2	4·6	4·34
7·0	6·0	5·83
13·2	11·2	10·90
27·1	22·4	21·48
58·8	44·8	45·80
94·0	67·2	70·02
136·0	89·6	95·60

$a = 0·839$ $b = 0·001$

H. Exp. I, 2.

x	y	y'
5·1	4·6	4·34
6·7	6·0	5.68
12·9	11·2	10·91
26·1	22·4	21·63
56·1	44·8	44·80
90·0	67·2	68·76
129·7	89·6	93·99
...	...	...

$a = 0·855$ $b = 0·001$

H. Exp. I, 3.

x	y	y'
3·8	3·2	3·32
5·2	4·6	4·52
7·0	6·0	6·06
13·3	11·2	11·45
27·6	22·4	22·76
59·8	44·8	45·47
95·8	67·2	65·95
138·8	89·6	83·60

$a = 0·880$ $b = 0·002$

H. Exp. I, 4.

x	y	y'
1·5	2	1·83
3·2	4	3·87
4·6	6	5·55
13·0	16	15·53
27·3	32	31·07
44·4	48	48·25
61·8	64	63·94
81·3	80	79·36
103·0	96	93·9

$a = 1·22$ $b = 0·003$

H. Exp. I, 5.

x	y	y'
2·5	5	3·78
4·5	7·5	6·79
6·5	10	9·79
13·4	20	20·12
27·0	40	40·59
58·0	80	81·43
89·5	120	119·98
122·4	160	156·04
158·5	200	189·76

$a = 1·52$ $b = 0·002$

H. Exp. I, 8.

x	y	y'
7	8	7·74
11	12	11·93
15	16	15·98
24	24	24·48
33	32	32·18
44	40	40·45
50	44	44·50
53	45	46·37
...	...	...

$a = 1·14$ $b = 0·005$

H. Exp. I, 9.

x	y	y'
7	8	7·88
10·2	12	11·35
14	16	15·38
22	24	23·43
31	32	31·90
40	40	39·82
51	48	47·55
62	56	56·11
...	...	...
...	...	...
...	...	...

$a = 1·158$ $b = 0·004$

H. Exp. I, 13.

x	y	y'
8·5	10·82	10·75
10·6	13·43	13·19
13·0	16·05	15·85
15·6	18·66	18·62
12·5	21·26	23·15
21·2	23·88	24·13
24·3	26·49	26·72
27·2	29·10	28·33
30·7	31·72	32·02
34·0	34·33	34·44
37·8	36·94	36·83

$a = 1·35$ $b = 0·01$

H. Exp. II, 1.

x	y	y'
3·3	2·2	2·2
6·2	4·2	4·2
12·0	8·0	8·0
24·0	16·0	15·8
37·0	24·0	23·8
51·0	32·0	32·3
64·9	40·0	40·3
79·8	48·0	48·4
95·3	56·0	56·5
112·0	64·0	64·8
131·0	72·0	73·5

$a = 0·679$ $b = 0·0009$

TABLE VII.—*Continued.*

H. Exp. II, 7.

x	y	y'
2·1	4	4·47
3·0	6	6·29
4·0	8	8·25
5·0	10	10·13
6·0	12	11·95
7·1	14	13·86
8·2	16	15·62
10·9	20	19·81
13·9	24	23·89
17·6	28	27·81
23·0	32	31·94
29·5	36	34·14
31·5	37	34·23
…	…	…
…	…	…
…	…	…
…	…	…
…	…	…
…	…	…

$a = 2·205 \quad b = 0·0355$

H. Exp. II, 8.

x	y	y'
7	8	8·04
10·5	12	11·96
12	14	13·56
14·5	16	16·22
18	20	19·84
22	24	23·81
26	28	27·65
31	32	31·86
41	40	40·65
45	44	43·75
51	48	48·10
56	52	51·47
62	56	55·20
	…	…
…	…	…
…	…	…
…	…	…
…	…	…
…	…	…

$a = 1·188 \quad b = 0·0048$

H. Exp. III, 3.

x	y	y'
4	8	7·8
8	16	15·4
12	24	22·9
17	32	31·9
22	40	40·7
26	48	47·5
31	56	56·0
36	64	63·8
42	72	73·0
47	80	80·4
52	88	87·5
58	96	95·6
64	104	103·4
71	112	111·9
79	120	121·0
85	128	127·4
96	136	137·9
105	146	145·6
116	154	153·6

$a = 1·974 \quad b = 0·0056$

H. Exp. III, 4.

x	y	y'
6·8	8·96	9·87
7·3	10·08	11·56
7·7	10·82	11·12
9·4	13·06	13·62
10·8	15·30	15·26
12·5	17·54	16·59
14·7	19·78	20·23
16·2	22·02	22·06
18·0	24·26	24·19
20·0	26·50	26·50
21·8	28·74	28·50
24·0	30·98	30·89
26·1	33·22	33·06
28·6	35·46	35·53
31·0	37·70	37·80
33·3	39·94	39·87
36·0	42·18	42·17
38·8	44·42	44·41
42·0	46·66	46·78
45·0	48·90	48·82
49·5	51·14	52·57
53·0	53·38	53·44

$a = 1·517 \quad b = 0·0096$

H. Exp. V, 1.

x	y	y'
31	22·1	22·50
68	45·0	45·37
114	67·0	67·67
141	78·0	77·70
171	84·6	84·90

$a = 0·776 \quad b = 0·0016$

H. Exp. VI, 1.

x	y	y'
28·0	32	31·18
61·2	64	64·11
100·0	96	97·00
122·0	112	113·00
146·0	128	120·50

$a = 1·17 \quad b = 0·002$

H. Exp. VI, 3.

x	y	y'
27·5	32	30·7
61·0	64	63·9
100·0	96·2	97·0
123·0	112	113·7
149·0	128	129·8

$a = 1·17 \quad b = 0·002$

F. Exp. I, 3.

x	y	y'
7·2	8	8·5
12·5	16	14·0
26·9	32	31·0
42·0	48	47·3
58·4	64	64·2
74·8	80	80·4
92·4	96	96·7
110·5	112	112·4
131·5	128	129·4

$a = 1·196 \quad b = 0·0016$

TABLE VII.—*Continued.*

F. Exp. I, 4.			F. Exp. II, 3.			F. Exp. II, 4.		
x	y	y'	x	y	y'	x	y	y'
2·8	4	3·98	3·1	4	3·66	3	4	3·70
6·0	8	8·11	7·0	8	8·00	6·6	8	7·90
9·2	12	12·10	10·9	12	11·98	10·3	12	11·95
12·5	16	15·99	15·2	16	16·04	14·4	16	16·11
16·2	20	20·46	20·0	20	20·20	18·8	20	20·21
20·3	24	24·24	25·1	24	24·05	23·8	24	24·39
24·2	28	27·94	30·7	28	27·66	29·0	28	28·22
29·0	32	31·84	34·3	30	29·52	35·5	32	32·18
31·6	34	33·78	...	...	...	39·0	34	33·78
$a=1·417$	$b=0·011$		$a=1·214$	$b=0·0102$		$a=1·263$	$b=0·01$	

F. Exp. III, 3.			F. Exp. III, 4.			F. Exp. IV, 3.		
x	y	y'	x	y	y'	x	y	y'
3·0	4	3·62	3·1	4	4·11	3·7	4	4·53
6·8	8	8·06	6·0	8	7·86	7·3	8	8·03
10·2	12	11·80	9·2	12	11·95	10·9	12	12·01
14·0	16	15·96	12·2	16	15·75	14·7	16	16·25
17·8	20	19·88	15·6	20	20·70	18·2	20	20·76
21·7	24	23·74	18·9	24	24·01	22·1	24	23·84
30·0	33	31·32	22·1	28	27·86	26·0	28	27·84
34·9	37	35·41	30·0	36	37·11	30·2	32	32·09
37·7	39	37·62	34·0	40	41·64	34·9	36	36·76
40·8	41	40·07	...	...	...	37·8	38	39·60
43·9	43	42·15	...	...	...	...	...	...
$a=1·224$	$b=0·006$		$a=1·327$	$b=0·003$		$a=1·123$	$b=0·002$	

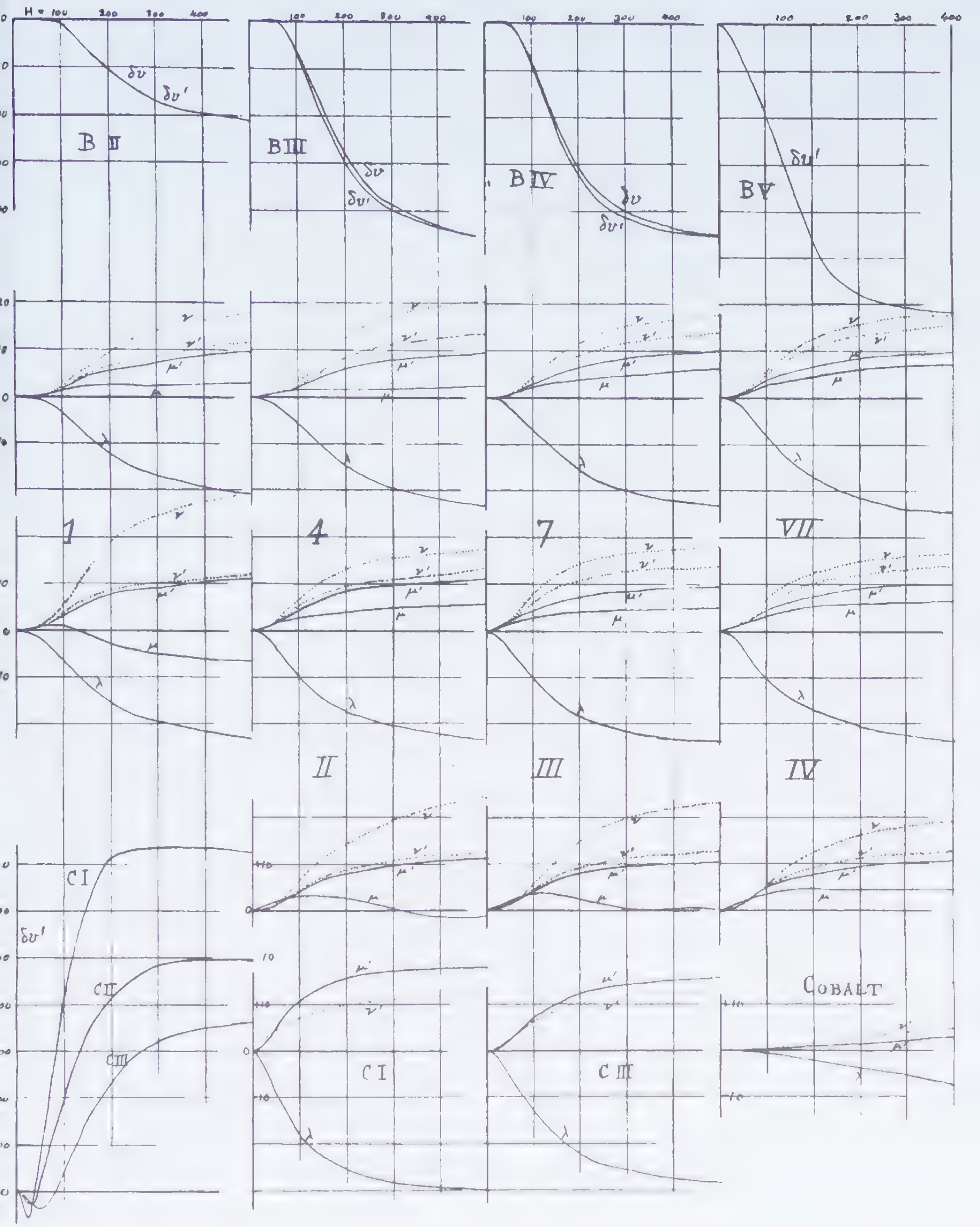
F. Exp. IV, 4.			F. Exp. V, 4.			F. Exp. VI, 4.		
x	y	y'	x	y	y'	x	y	y'
3·5	4	3·81	3·2	4	3·97	3·3	4	3·86
7·0	8	7·73	6·6	8	7·79	7·0	8	7·91
10·8	12	11·81	10·1	12	11·60	11·0	12	11·99
14·6	16	15·79	14·1	16	16·10	15·3	16	16·02
18·3	20	19·58	18·1	20	20·04	20·0	20	20·00
22·0	24	23·41	22·9	24	24·37	25·0	24	23·75
26·1	28	27·25	27·6	28	29·20	30·6	28	27·36
30·4	32	31·53	33·0	32	32·11	37·2	32	31·27
32·8	34	33·69	35·5	33	33·75	...	...	...
35·2	36	35·89	...	...	...	...	...	...
38·0	38	38·42	...	...	...	...	...	...
$a=1·125$	$b=0·003$		$a=1·27$	$b=0·009$		$a=1·2$	$b=0·01$	

TABLE VII.—*Continued.*

F. Exp. VII, 3.			F. Exp. VIII, 4.			F. Exp. IX, 2.			F. Exp. X, 1.		
x	y	y'	x	y	y'	x	y	y'	x	y	y'
7·0	8	8·43	3·2	4	3·66	6·7	8	7·98	7·9	8	8·19
13·8	16	16·23	6·1	8	7·92	13·3	16	15·99	16·0	16	16·33
27·0	32	31·52	9·7	12	11·97	26·7	32	30·75	32·6	32	32·31
42·2	48	47·82	12·8	16	16·22	42·1	48	48·19	50·7	48	48·24
58·7	64	64·60	16·1	20	19·95	58·9	64	64·04	70·0	64	64·78
74·9	80	80·02	19·8	24	24·11	76·7	80	80·68	90·9	80	79·19
92·8	96	96·78	23·2	28	27·88	96·1	96	97·33	114·1	96	94·08
112·2	112	111·40	27·3	32	32·00	117·7	112	113·90	142·8	112	109·60
...	...	...	29·3	34	33·99	142·0	128	130·78	157·1	118	116·00
	...	...	31·8	36	35·43	155·4	136	136·94	...	...	...
...	...	...	34·0	38	38·49	...	...	...	...	...	...
$a=1{\cdot}218$ $b=0{\cdot}002$			$a=1{\cdot}336$ $b=0{\cdot}006$			$a=1{\cdot}205$ $b=0{\cdot}002$			$a=1{\cdot}053$ $b=0{\cdot}002$		

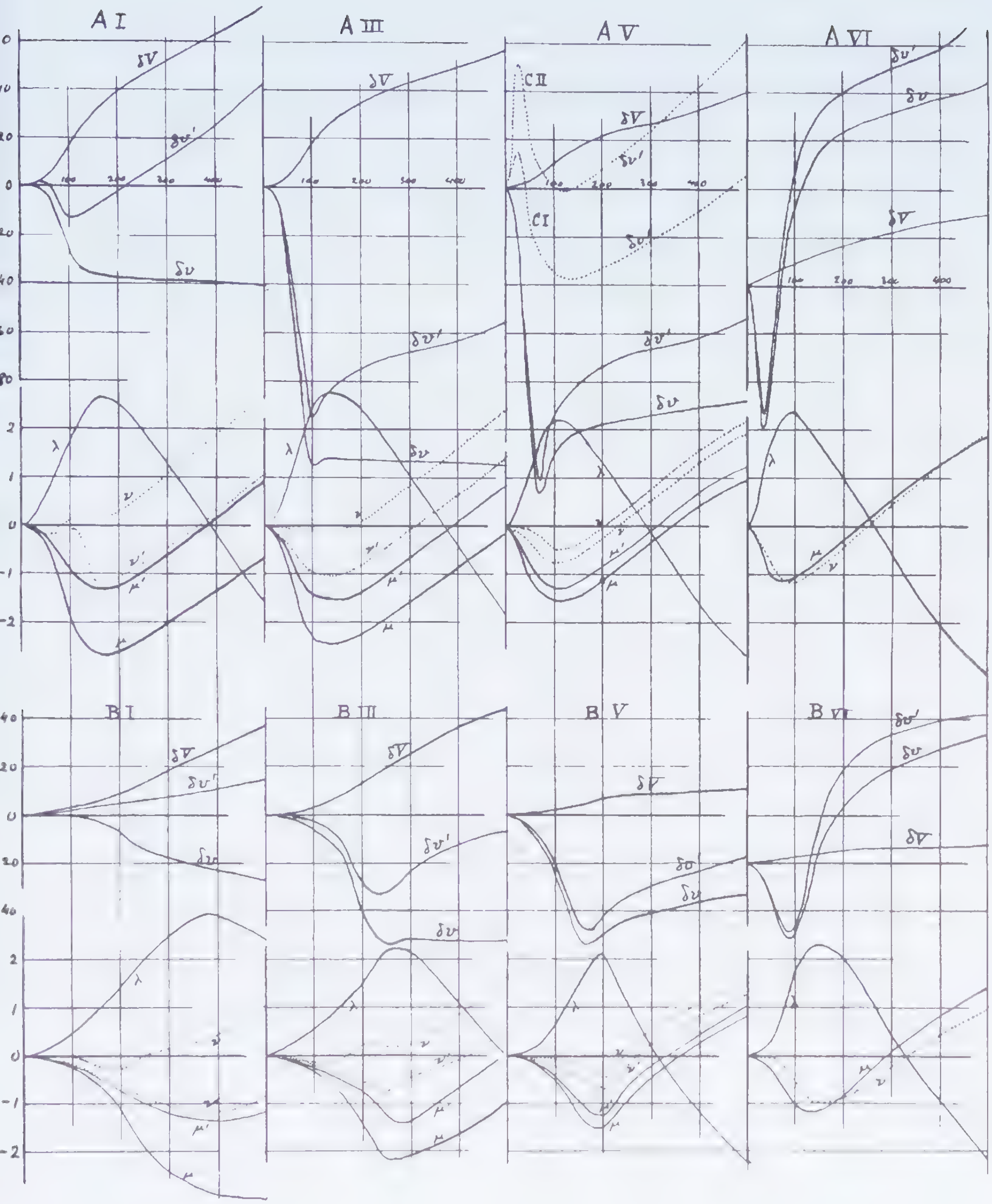
NICKEL TUBES AND COBALT TUBE.

PLATE I.



IRON TUBES, A, B, AND C.

PLATE II.



XV.—*The Strains produced in Iron, Steel, Nickel, and Cobalt Tubes in the Magnetic Field.* Part II. By Professor C. G. Knott, D.Sc., F.R.S.E. (Plates I. and II.)

(Read 6th June 1898.)

CONTENTS.

§ 1. Introduction.—The remarkable changes produced by magnetization in the internal volumes of hollow cylinders of iron, steel, and nickel have been described in Part I. (see *Trans. R.S.E.*, vol. xxxviii. pp. 531–555). As pointed out in the closing paragraph, a complete discussion of these changes had to be "deferred until direct measurements of elongation had been obtained with the various tubes under the same magnetic influences." It was not possible, of course, to measure the elongations of *all* the tubes that had been experimented with; for of these, eighteen (Nos. I. to VI. of each inclusive) were no longer in existence, having been the successive stages through which No. VII. was brought from the condition of small bore and thick walls to that of wide bore and thin walls.

As the investigation with the existing tubes proceeded, it became more and more matter for regret that the idea of measuring the elongations as well as the volume changes of the successive tubes had not occurred to me at an earlier stage. I therefore resolved to carry out a complete series of experiments with a new set of iron tubes, all successive stages in the life-history of one and the same bar. These are distinguished below as Nos. I′. to VIII′. inclusive. The changes of length and the changes of volume of bore of each of these tubes were measured, and from these measurements certain interesting results were obtained.

But it now became evident that a much clearer insight into the character of the strain accompanying magnetization in iron and nickel tubes would be obtained if the cubical dilatation of the *material* of these tubes could be measured directly. This, unfortunately, could not be effected with the tubes in use, which nearly filled the core of the magnetizing coil (Part I., § 6).

Led by these considerations, I proceeded to study the volume and length changes of sets of smaller tubes, each of which could be enclosed in a strong brass tube inserted in the core of the magnetizing coil. The new nickel tubes, distinguished as the B tubes, were formed by successive borings from a nickel bar, 20·2 cm. long and 2·72 cm.

in diameter. Two new sets of iron tubes, distinguished as the A tubes and the B tubes, were studied. The B tubes correspond in dimensions with the nickel B tubes; the A tubes are twice as long. Each original iron bar gave in succession seven tubes, the internal diameters of which increased from two-eighths of an inch (No. I.) to fully an inch (No. VII.). The internal diameters of the nickel tubes range from three-eighths (No. II.) to six-eighths (No. V.) of an inch. A further boring was in this case out of the question, since the material had begun to crack, and the tube consequently to leak. The dimensions of these and the other new tubes are given in numerical detail in Table I. at the end of the paper.

Having thus briefly sketched the history of the research, I propose in what follows (1) to discuss in full the results for the A and B tubes, and (2) to compare with these, and elucidate by their means, the comparatively incomplete results obtained with the large tubes, both old and new.

The existing old tubes are No. VII., 3, 5, 7 of iron and of steel; and No. VII., 1, 4, 7 of nickel.

The iron and steel tubes, No. 9, are the thinnest walled of all the large tubes, and differ from the others in having no internal ledge on which a washer could be screwed down under the cap. The measurement of the volume changes in these thin tubes required a different method of fitting the cap, and, indeed, a different cap altogether. The results originally obtained with them were not regarded as altogether satisfactory, and were accordingly omitted in Part I. They are given below, for the sake of completeness, along with the elongations.

Excluding the eighteen temporary tubes which formed the successive stages of the iron, steel, and nickel tubes, No. VII., we have in all forty tubes, whose changes of form and volume in various magnetic fields are now to be discussed.

§ 2. METHODS OF EXPERIMENT.—With each of the A and B tubes four distinct experiments were made. These were:—

(1) Measurements in various magnetic fields of the corresponding changes of volume of bore.

(2) Measurements in the same fields of changes of length.

(3) Measurements in the same fields of changes of volume of the *material* of the tube.

(4) Measurements in the same fields of *apparent* external changes of volume, the tube being plugged and treated as a bar.

The first form of experiment was conducted after the manner described in Part I., § 7. The metal tube and the connected capillary glass tube were filled with water, and the changes of volume measured by the displacements of the end of the water column in the capillary.

In the second form of experiment the change of length was measured by means of a lever and mirror arrangement, similar in essence to the arrangements used by other experimenters (JOULE, BARRETT, BIDWELL, etc.). The tube under investigation rested

inside the brass tube already mentioned, which was adjusted within the magnetizing coil so that the iron or nickel tube was centrally placed in the coil. The mirror rested by two colinear knife-edges on supports firmly attached to a brass cap screwed to the top of the brass tube. A glass rod of suitable length rested by its lower end on the top of the iron or nickel tube, and supported on its upper end a third knife-edge also fixed to the mirror. This knife-edge was parallel to, but lay 1·1 mm. behind, the common line of the other two knife-edges. Any change of length in the inner tube would produce a rise or fall of the support of the single knife-edge, while the two colinear knife-edges would be unaffected. The consequent tilt given to the mirror, which was set approximately vertical, was measured by means of a telescope and reflected scale in the usual manner. A simple calculation gave the corresponding change of length of the tube, and from that the longitudinal dilatation could at once be found.

Experiments (1) and (2) were made with the large tubes also. In the measurement of the change of length, however, the method was slightly modified. The brass cap supporting the colinear knife-edges was screwed on to the top of the iron, steel, or nickel tube, while the glass rod supporting the single knife-edge passed up through the hollow core from the base of the tube.

In the third form of experiment the iron or nickel tube was placed within the brass tube, which was otherwise filled with water, and to which the capillary was attached. Any change of volume of the material of the immersed tube produced its effect on the position of the end of the water column in the capillary.

In the fourth form of experiment the iron or nickel tube was plugged up (air only being inside), and in this condition was dropped into the brass tube, while everything else was exactly as in experiment (3).

Unless the differences in the mechanical constraints to which any tube was subjected in these various experiments produce really important disturbances, we should expect to find the *apparent* volume change of experiment (4) to be equal to the algebraic sum of the volume changes of experiments (1) and (3).

A glance at the numbers given in Tables II. and VI. below will show how satisfactorily the experiments establish this relation.

§ 3. THE STRAIN COEFFICIENTS.—In these experiments the quantities directly measurable are :—

- V, the volume of the material ;
- v, the volume of the bore ;
- v', the volume of the space enclosed by the outer surface and ends—in other words, the volume of the original bar from which the tubes were formed ;
- δV, δv, $\delta v'$, the changes of these volumes in given fields ;
- λ, the longitudinal dilatation of the tube ; and
- δ, $= \delta V/V$, the average cubical dilatation of the material of the tube in these same fields.

Now we may write

$$\delta v/v = \lambda + 2\mu \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

where μ represents the transverse dilatation of the core. It may be regarded as measuring the elongation at each point of the inner surface of the tube in the direction perpendicular to the axial plane passing through the point. If we suppose δ to be the cubical dilatation at this point, we have the equation

$$\delta = \lambda + \mu + \nu \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where ν is the elongation in the direction of the radius.

Similarly, $\frac{1}{2}(\delta v'/v' - \lambda)$ gives μ', the "tangential" elongation at the outer surface of the tube; and then the radial elongation is given by

$$\nu' = \delta - \lambda - \mu' \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

In calculating these strain coefficients I make the two assumptions:—First, that δ, which really measures the average cubical dilatation throughout the metal, also measures the cubical dilatations of the elements at the surfaces; and second, that λ has the same value at every point of the tube. There seems no way of testing the truth of the first assumption; but the second was tested by direct experiment, and no indication was found of λ having different values at the outer and inner walls.

The precise significance of the ratios λ, μ, ν, μ', ν' may be thus indicated. Imagine a small spherical element of diameter $2e$ at the inner surface of the tube. After the application of the magnetic stress this sphere becomes an ellipsoid, whose principal axes are $2e(1+\lambda)$ in a direction parallel to the axis of the tube, $2e(1+\nu)$ in a radial direction, and $2e(1+\mu)$ in a direction at right angles to these—that is, tangential. The ratios $1+\lambda$, $1+\nu'$, $1+\mu'$ have similar meanings for an originally spherical element at the outer surface.

Again, if r, R, are the inner and outer radii of the tube, $r\mu$ and $R\mu'$ represent the outward displacements of the corresponding surfaces.

Although it is not possible to calculate accurately these strain coefficients in the case of the large tubes, we may obtain an approximate estimate of their values on the further assumption that the cubical dilatations are the same for all tubes of the same metal. Thus, since $V + v = v'$, we have $\delta V + \delta v = \delta v'$ in any given field. Hence

$$\frac{\delta V}{V} + \frac{\delta v}{v}\,\frac{v}{V} = \frac{\delta v'}{v'}\,\frac{v'}{V}$$

or,

$$\frac{V}{v'}\left\{\delta + (\lambda + 2\mu)\frac{v}{V}\right\} = \lambda + 2\mu' \quad . \quad . \quad . \qquad (3)$$

Now, in the case of the large tubes, λ and $\lambda + 2\mu$ are measured by direct experiments, and the volumes v, v', V are known. Consequently, if δ be assumed, the value of $\lambda + 2\mu'$ at once follows. Hence μ and μ' may be calculated. The values of ν and ν' are then found from the equations (2) above.

This process has been applied to the old iron tubes 1, 3, 5, VII., to the new iron tubes I′. to VIII′., and to tube A VII.

As will be seen immediately, the cubical dilatation in the case of nickel is negligibly small compared with the elongations, so that μ' is to be found from the simplified form of equation

$$\lambda + 2\mu' = \frac{v}{v'}(\lambda + 2\mu) \; . \qquad . \qquad . \qquad . \qquad . \qquad (3)'$$

The ratios ν and ν' follow as above.

The values for the Nickel Tubes 1, 4, 7, and I. to VII. have been obtained in this way, the additional assumption being made that the elongation λ is the same for the non-existent Tubes I. to VI. as for the final existent form VII. A consideration of all the measured values of λ for the various tubes, large and small, will show that this assumption, though not strictly true, does not involve an error of magnitude sufficient to modify seriously the final conclusions.

§ 4. THE BORED NICKEL TUBES.—These are best considered first, because of the comparative simplicity of the results obtained. The volume changes and dilatations of the B tubes in various magnetic fields are given in Table II., and are shown graphically in the first two rows of Plate I.

Especially noteworthy is the smallness of the material volume change in comparison with the other measured volume changes. So minute is δV, that in calculating the strain coefficients we may, without any risk of serious error, put

$$\lambda + \mu + \nu = 0.$$

The δv and $\delta v'$ curves lie very close together. It is, in fact, hardly possible, on the chosen scale, to draw them distinct in the case of B II. For B V. one curve only is given, that, namely, which shows how the apparent external volume change increases with the field. A tiny crack in the wall of the tube prevented any good observation of the bore change being made.

On the whole, there is a tendency for the volume changes δv and $\delta v'$ to differ more as the bore increases, that is, as the walls get thinner. This may be referred to the different conditions of constraint in the two forms of experiment. When δv was being measured, the brass cap, by means of which the capillary was attached to the upper end of the tube, was screwed on to the outside surface. On the other hand, in order to permit the tube to slide easily within the brass tube when $\delta v'$ was to be measured, the nickel tube was in this case closed by a brass plug which screwed on to the inside surface. Thus in any lateral expansion of the tube, there would be more constraint with the outside fitting cap than with the inside fitting plug.

Again, the manner in which the cubical dilatation diminishes with the thickness of the wall suggests the possibility that part of the measured change of volume δV may be due to empty spaces within the metal—in other words, to its vesicular structure.

Passing to the consideration of the coefficients of strain, we notice a steady, though

small numerical increase in the values of the longitudinal elongations (λ) as the bore of the tube is increased—that is, as the thickness of the wall is diminished. This is quite in accordance with what might be expected if the elongation depends on the induction rather than on the field. For, as I found by direct experiment in the case of the large tubes, the induction in a given field is smaller in the tube of narrower bore or wider wall.

The "tangential" elongations (μ) at the inner surface are all positive and much smaller numerically than the longitudinal elongations. They show a tendency, however, to increase with the bore. Because of the comparatively small value (practically zero) of the cubical dilatation, the corresponding "radial" elongations (ν) are distinctly larger than the tangential elongations, but show no marked tendency either to diminish or increase as the bore varies.

On the other hand, the tangential and radial elongations (μ', ν') at the outer surface behave somewhat in contrary fashion, μ' increasing, and ν' correspondingly decreasing, as the bore increases. For any given field and tube the four ratios μ, μ', ν', ν are in order of magnitude, μ' and ν' approximating to equality when the bore is narrow, and gradually diverging in value as the bore increases. These relations are well shown in the curves.

Had it been possible to obtain wider bores without hopelessly damaging the tube, which already showed signs of cracking, it is highly probable that μ' and ν' would again approximate to equality, just as was found to be the case with the like quantities for the iron tube (see below).

By application of equation (3)′ of last paragraph, the ratio μ' was calculated for the large Nickel Tubes 1, 4, 7, and VII. Similar calculations were also made for Tubes I. to VI., the elongation λ being assumed to be the same for these as for their final form VII. Although this assumption is not strictly accurate, it is sufficiently near the truth not to lead to any serious error in the calculations of the other ratios.

The results for the four existing tubes are given in Table III., and are represented graphically in the third row of Plate I.

An epitome of the results for Tubes I. to VII. forms Table IV., and some of the features are shown graphically in the fourth row of Plate I.

The volume changes measured are, of course, much larger for these tubes than for the B tubes. Nevertheless, the linear dilatations come out with values practically identical in the two sets. This will appear the more remarkable when the different conditions under which the large and small tubes are magnetized are borne in mind. Each large tube when inserted in the magnetizing coil extended to within a few inches of each end, and could not therefore be magnetized so uniformly as one of the short tubes which lay much more completely within the magnetizing coil.

One interesting point brought out by the large tubes is the negative value of μ in the tubes of narrowest bore under high magnetizing forces. See, for example, the μ-graphs of No. 1, II. and III. (Plate I.). Also there is an interesting gradation in the effect as the bore increases. Thus the values of μ for Nos. 1 and I. are positive in fields lower than 140 and 180 respectively, and negative in higher. In Tube No. II.

the same feature is shown, but the change of sign occurs about Field 270. In Tube III. the *form* of the graph is the same, but no negative value is reached. In Tube IV., however, this characteristic has disappeared, and the behaviour is, broadly speaking, the same as in Tube VII. Notwithstanding this peculiarity in the sign of μ in the narrow-bored tubes, the calculated values of μ' ν' come out nearly the same for all. It will be readily seen, both from the curves and from Table IV., that the comparative values of μ' ν' follow the same law of change as in the B tubes, approximating in value in the tube of narrowest bore, and gradually drawing apart as the bore increases.

§ 5. The Coiled Nickel and Cobalt Tubes.—These are formed from sheets of metal, each tube being about 10 inches long and 1 inch diameter. After the long edges had been soldered together to form a hollow tube, the changes of length and the changes of volume of the material in various fields were measured, as already described. The tube was then plugged up with brass discs at both ends, and measurements were made of its changes of bulk.

From these measurements the dilatations λ, μ', ν' follow at once. It seems hardly necessary to trouble calculating μ and ν in these cases of very thin-walled tubes. A glance at formula (3), § 3, shows that the comparative smallness of V makes the quantities μ, ν differ very slightly from μ', ν'.

The three Nickel Tubes C I., C II., C III. were formed from three sheets of different thicknesses. The dimensions are given in Table I. The nickel was obtained as pure as possible, and in this respect is better than the nickel of the bored tubes, which contains $2\frac{1}{2}$ to 3 per cent. of impurities. This may account for the fact that the longitudinal contraction is distinctly greater in the C tubes than in any of the others.

It is, however, in the external volume changes that the greatest difference is shown between the coiled tubes of very thin wall and the bored tubes of comparatively thick wall. With all of the C tubes there is *increase* of external volume except in the lowest fields; in every other instance *decrease* was the characteristic feature. Compare, for example, the Curves C I., C II., C III. in the lowest left-hand corner of Plate I. with the curves in the first row and with similar curves in the former paper.

It is interesting to note that the volume increase is greatest in the tube of thinnest wall, and falls off as the wall is made thicker. With a still thicker wall the volume change might change sign and become negative. It would not be safe, however, to institute any strict comparison between tubes formed by coiling sheets and tubes formed from solid bars by boring.

Excepting that the μ' curve lies higher than the ν' curve, there is not any great diversity shown in the nature of the linear dilatations in the various types of tube. The C I. and C III. curves, occupying the middle of the last row in Plate I., are very similar to the λ, μ', ν' curves in the third and fourth rows of the same plate. It may be mentioned that C II. differs so very little from C III. as hardly to require a separate set of curves. The greater divergence between the values of μ' and ν' in C I. than in either of the others is noteworthy as being somewhat unexpected.

But the differences that exist seem of comparatively small importance beside the general agreement, even to numerical details, among the dilatations of the different types of nickel tube.

The results for the cobalt tube call for little remark. The linear dilatations are much smaller than for nickel, as a glance at the curves on Plate I. shows. The cubical dilatation of the metal is, however, greater, being appreciable enough to be measured.

The broad difference between the two metals is that the molecular groups yield more readily to the magnetizing force in nickel than in cobalt. The nickel curves all show an approximately saturated condition in the substance; but there is no evidence of such a condition being approached in the case of cobalt.

§ 6. The Bored Iron and Steel Tubes.—The results for iron are, as compared with those for nickel, of a very complex character.

Consider, first of all, the volume changes of bore in the various sets of tubes as given in the column headed δv in Tables VI., VIII., IX. A gradual change in the behaviour of the tube as the bore is made larger is very apparent in the case of Tubes A and B (Table VI. and Plate II., first and third rows of graphs). Also there is an evident parallelism in the two series A and B. Thus, in I. and II. of both sets, the volume of bore diminishes steadily as the field increases. In III., however, the change of volume passes through a curious minimum distinctly shown in the graphs. In IV. and V. this minimum becomes more evident, the negative change of volume diminishing markedly in the higher fields. Finally, in VI. and VII. this diminishing negative change becomes an increasing positive change. In the A set a particular peculiarity is associated with a field which is lower than that associated with the corresponding peculiarity in the B set. For example, the fields associated with the minimum volume change or greatest *diminution* of volume are :—

In	A III.	105	and	260	in	B III. ;
„	A IV.	90	„	200	„	B IV. ;
„	A V.	70	„	165	„	B V. ;
„	A VI.	40	„	90	„	B VI. ;
„	A VII.	20	„	40	„	B VII. ;

while the change of sign from negative to positive volume change occurs in fields

70, 35, 140, and 75

in A VI., A VII., B VI., and B VII. respectively.

A similar parallelism is shown in the two sets of results for the changes in external volume of the plugged tubes. Compare, for example, the $\delta v'$ curves in the first and third rows of Plate II.

The same feature is reproduced in the columns headed λ, the measurements, namely, of the linear dilatation in the direction of the magnetizing force. A glance at the curves figured in the second and fourth rows of Plate II. shows

that the maximum value of λ in each A tube occurs in a lower field than it does in the corresponding B tube. Thus the field of maximum λ is :—

In A	I.	165	and	370	in B	I.
„ A	II.	150	„	310	„ B	II.
„ A	III.	140	„	270	„ B	III.
„ A	IV.	125	„	225	„ B	IV.
„ A	V.	110	„	190	„ B	V.
„ A	VI.	95	„	150	„ B	VI.
„ A	VII.	80	„	100	„ B	VII.

Thus the B tubes reproduce, but in higher fields, the peculiarities shown by the A tubes. The reason is not far to seek. It depends on the fact that the shorter B tubes have a larger *demagnetizing factor* than the larger A tubes. Not only so, but, in accordance with well-known results, the demagnetizing factor diminishes with the area of section of the material, the length being constant. This consideration explains at once the gradual shifting of the critical points (maximum λ, minimum δv and $\delta v'$) into lower fields as the bore of a tube of given length is gradually increased.

The shifting of the crest of the longitudinal elongation curve as we pass through the series A I. to A VII. and B I. to B VII. is also well shown with tubes I'. to VIII'. An inspection of Table IX. will bring this out clearly enough; but the feature is most distinctly shown in the following table constructed from the curves corresponding to Table IX., which curves, however, being broadly similar to those in Plate II., I have not thought it necessary to publish.

FIELDS FOR MAXIMUM ELONGATION IN IRON TUBES I'. TO VIII'.

Tube.	I'.	II'.	III'.	IV'.	V'.	VI'.	VII'.	VIII'.
Field	230	215	200	180	170	155	140	120

According to the commonly accepted theory, the demagnetizing factor in a cylindrical bar is proportional to the square of the ratio of the diameter to the length. In the case of a cylindrical *tube* this law requires modification. Perhaps the most probable *simple* hypothesis is to compare the tube, as regards its demagnetizing factor, to a bar of the same length and the same cross-section of material. How far this applies to the present case is tested immediately by a comparison of Tubes A II. and B VI., which have their maximum elongations in about the same field. Presumably their demagnetizing factors are nearly the same. Now the cross-sections of the material of A II. and B VI. are as 5·96 to 1·94. Dividing these respectively by 4 and 1, which are as the squares of the lengths of the A and B tubes, we get for the ratio of the demagnetizing factors 194 : 144 or

27 : 20. The hypothesis suggested above requires this ratio to be unity. In making this comparison, however, we should bear in mind that the B tube is, because of its shorter length, more uniformly magnetized throughout, and has therefore less leakage of lines of force from the sides than the A tube. This would tend to accelerate the shifting of the maximum to lower fields, an effect which in less uniform fields would be produced by widening the bore of the tube.

The cross-sections of A V. and B VII. are as 3·24 to ·82; hence their demagnetizing factors are as 81 to 82. These tubes should, according to the theory, have their maximum elongations in the same field. As a matter of fact, the field corresponding to the maximum elongation of A V. is somewhat higher than that of B VII.

A similar comparison of Tubes VII′. and A III., in both of which the field of maximum elongation is 140, brings out 39 : 32 as the ratio of the demagnetizing factors. The deviation of this ratio from unity may also be partly accounted for by the greater uniformity of magnetization in the somewhat shorter A tube.

The steadiness with which in all cases the field of maximum elongation diminishes as the bore is widened is particularly noteworthy.

Contrary to the usual experience, I have had no difficulty in measuring the changes of volume of the magnetized material. A glance at the δV columns of Table VI. shows that these changes in the case of iron are by no means insignificant. The corresponding dilatations (δ in Table VII.) have fairly similar values in the A tubes until A VII. is reached, but vary considerably from tube to tube in the B series.

The tendency is for the cubical dilatation to diminish as the walls of the tube become very thin. Also, both in A VII. and B VII. there is a maximum dilatation in a moderate field. The existence of this maximum need in no way surprise us, for a like peculiarity appears in the longitudinal dilatation. It is rather matter for surprise that there should have been no hint at a maximum cubical dilatation with the other tubes. I have already suggested that the cubical dilatation may be appreciably affected by a vesicular condition in the material, and it is well known that such molecular changes are influenced by the form of the body. Now the thinner the wall, the less chance is there of the presence of vesicular cavities; and it is also conceivable that in very thin walled tubes there is increased uniformity of magnetization with possibly less leakage of the lines of induction from the sides. Either or both of these considerations seem to give a sufficient explanation of the phenomenon just described.

§ 7. Curious Behaviour of Iron Tubes under Certain Conditions.—In all experiments in which the iron tubes were enclosed in the brass tube—that is, in experiments of type (3) and (4) of § 2—a curious effect was observed, of which, so far, I have been unable to find a satisfactory explanation. I can best indicate its nature by transcribing from the experimental note-book the unreduced numbers which measured the volume changes. These represent the number of divisions on the micrometer scale through which the water meniscus in the capillary tube appeared

to move at the instant the magnetic field was established or removed. The positive sign means that the meniscus moved outwards along the capillary, showing an increase of volume of the iron tube contained in the brass tube. The negative sign means, of course, a movement in the opposite direction, indicating a decrease in the bulk of the contained iron tube. When two numbers are entered side by side in the third column, the first (in brackets) gives the greatest excursion in the indicated direction made by the meniscus before it comes to its final position of rest after removal of the magnetic field.

Iron Tube A III., November 12th, 1897.
Apparent Change of External Volume; Tube Plugged.

Field.	Field on.	Field off.
531	− 25	(− 23) + 11
403	− 29	(− 13) + 19
302	− 30·5	(− 11) + 22
...	...	... ...
202	− 33	(− 8) + 27
...	...	... ...
100	− 42	(− 1) + 38
...	...	... ...

A III., November 15th, 1897.
Change of Volume of Metal.

536	+ 18	− 34
408	+ 15	− 28
308	+ 14	− 24
...	...	...
100	+ 5·5	− 8·5
...	...	...

Thus, taking the very first result given above, we see that, when a field of 531 was established, the water meniscus in the capillary moved back 25 divisions of the micrometer scale, and there came to rest. On removal of the field, the meniscus darted back 23 other divisions, then turned and moved forward along the capillary, stopping finally + 11 divisions from the position it occupied just before the field was removed. Its final position was therefore $-14 (= -25 + 11)$ from that originally occupied just before the field was established.

Again, taking the first result of the experiment of November 15th, we see that the establishment of Field 536 produced a forward motion of the meniscus through 18 divisions of the scale, but that, on the removal of the field, the meniscus moved back through 34 divisions, occupying a final position -16 from that occupied before the field was established.

The results are the same for both directions of field. In every case the negative reading is greater than the positive. In the experiment of November 12th, the

phenomenon has a strong resemblance to hystèresis; but, as proved by the other experiment, in which the return value is the greater, it is merely a resemblance and nothing more. That the phenomenon does not depend only on the iron, but has something to do with the brass tube, is proved by the fact that there is not the slightest evidence of its existence in the experiment for measuring the change of volume of bore. Thus, compare with the foregoing tables the following :—

A III., November 12th, 1897.

CHANGE OF VOLUME OF BORE.

Field.	Field on.	Field off.
531	+47	−47·5
403	+47	−47·8
302	+46·5	−47
...	...	...
100	+47	−47
...	...	...
48	+14	−13

In this case the capillary was in connection with the interior of the tube, and a positive reading means a contraction of the volume.

The complete absence of the peculiar effect in this last case quite disposes of any attempt at explanation in terms of hysteresis or possible change of temperature. The effect is as if at every make and break of the current in the magnetizing coil the brass tube containing the iron were permanently increased in internal capacity, or as if a certain quantity of water in the tube were removed from it. It is conceivable that a sudden increase of volume of the inclosed iron might push out a small quantity of water so that a positive reading might be followed by a greater negative reading, as in the experiment of November 15th; but it is altogether inconceivable that a contraction of the iron should be accompanied by a like pushing out of water, as would have to be the case were the explanation to hold good for the first experiment of November 12th.

One early suggestion was that the slight difference of diameter between the iron tubes and the bore of the brass tube might produce a certain constraint on the film of water between. But it was found on trial that the peculiar effect persisted in the case of an iron bar whose diameter was made distinctly smaller than the diameter of bore of the brass tube.

Thus, by a process of exclusion, we seem to be driven to the view that the brass tube must experience, when the field is established, an abrupt change of volume from which it does not immediately recover, or from which it recovers slowly when the field is removed. Care was taken to have the iron tube as nearly as possible centrally placed in the magnetizing coil. But no doubt there was some lack of perfect symmetry, so that,

when the iron tube became magnetized, extra pressures between it and the bottom or walls of the brass tube might very easily come into existence; and these might reasonably enough be supposed to produce minute but appreciable changes of volume. In one experiment the tube was set 2 inches above the usual position in the magnetizing coil; but this displacement from the central position did not to any decisive extent affect the readings—a point which rather tells against the explanation just given. But the most serious difficulty is the magnitude of the strain, which, though small, represents stresses of considerable magnitude. The micrometer-reading 14 means a volume change of 34×10^{-6} cub. cm. The bore of the brass tube was 46 cms. long and 3 cms. in diameter, giving a cubical dilatation of fully 10^{-7} within the tube. We may get an approximate idea of the pressure required to produce this dilatation by solving the equation

$$10^{-7} = \frac{Pa^2 - P'b^2}{b^2 - a^2} \; \frac{1}{k}$$

where P P′ are the internal and external pressures, a and b the inner and outer radii of the tube, and k the reciprocal of the compressibility. In the present case b is 2·1 cms., k may be put equal to 10^{12} C.G.S. units, and P′ is the atmospheric pressure, say 10^6. Thus—

$$Pa^2 = 10^6(4{\cdot}41 + {\cdot}22)$$

and

$$P = 10^6 \times 2{\cdot}06\,.$$

Hence it would need an *additional* pressure within of fully an atmosphere to produce the dilatation observed. This seems hardly possible under the conditions, for the interior communicates with the outside through the capillary.

The extraordinary manner in which the plugged Tube A III. passed through its changes when the field was removed was common to the tubes from A I. to A IV., and from B I. to B III. But there was no evidence of it with A V., B IV., and the tubes of wider bore than these. In all cases in which it occurred, the effect was the same—an increased negative change at the instant the magnetizing current was broken, followed immediately by a return to a final positive or diminished negative change. The phenomenon may be explained as an effect of the current of self-induction, producing a short-lived but intense magnetization in the *outer* layers of the iron tube, for the phenomenon is not observed in the experiments for measuring the change of volume of bore—that is, in experiments on the behaviour of the tube at the inner wall. If this, however, were the sole cause, it is hardly likely that it would so entirely disappear in A V. and B IV., which still have fairly thick walls. It is conceivable that there may also be a tendency for the transverse dilatation μ' to disappear more quickly than the longitudinal dilatation λ. A slight tendency in λ to persist when the field was removed—in other words, an appreciable time-lag—would, in the higher fields, give rise to a momentary increase in the diminution of volume.

The quantities tabulated under δV and $\delta v'$ are the means of the readings obtained on application and withdrawal of the magnetizing force. In the majority of cases, the sum $\delta V + \delta v$ is not very different from $\delta v'$, a result which gives a check on the accuracy of the experiment.

§ 8. The Dilatations in Iron and Steel Tubes (Bored).—From the volume and length changes, in the manner described in § 3, the dilatations δ, λ, μ, ν, μ', ν' were calculated, and are entered in the appropriate columns in Table VII. A selection of the results is given graphically in the second and fourth rows of Plate II.

There is a tendency for the dilatations μ and ν to be of opposite sign from λ—that is to say, they are, as a rule, negative when λ is positive, and positive when λ is negative. The curves obtained by plotting the dilatations in terms of the magnetic fields are, with one exception, remarkably smooth, and in no way reflect the peculiar features of the δv and $\delta v'$ curves, from which they have been derived by a purely arithmetical process. Compare, for example, the groups of curves for A V., A VI., B V., and B VI. The grouping of the λ, μ, ν curves is broadly the same in all; and yet, at first sight, nothing is more striking than the differences between the δv and $\delta v'$ curves for V. and VI. The reason for this is not far to seek. The ratios $\delta v/v$ and $\delta v'/v'$ become for the thinner walled tubes distinctly smaller than λ, so that the calculated values of μ and μ' are more affected by the peculiarities of λ than by those of $(\lambda + 2\mu)$ and $(\lambda + 2\mu')$.

One interesting feature in regard to the relative magnitudes of μ and ν is that they, so to speak, change places in the transition from Tube A V. to A VI., and from B VI. to B VII. In other words, we find, on plotting the curves, that the μ curves lie below the corresponding ν curves up to A V. and B VI., and that thereafter the μ curves lie above the ν curves. In this respect the coiled tubes discussed in next paragraph are comparable to A VII. and B VII. This result seems to be one that could hardly have been expected. A gradual convergence to equality might more reasonably have been looked for, as the tube was taken thinner and thinner. The interchange of μ and ν in regard to magnitude is, however, characteristic of both the nickel and the iron, and must have some fundamental significance.

The results for the other iron tubes given in Tables VIII. and IX. are very similar to those for the A and B series. In these tables an average value for δ is assumed, so that the values of ν, μ', and ν' are, at the best, tentative. It is, however, noteworthy that the large Tubes 3, 5, 7, VII., and 9, and I'. to VIII'. should give results, broadly speaking, identical with those obtained from the shorter and narrower tubes of the A and B series. It is instructive to note how comparatively little the values of the transverse and radial dilatations are affected by the sign of the change of volume of bore. As already pointed out, the longitudinal dilatation λ has a preponderating influence upon the character of the other dilatations. To express it otherwise, the ratio $\lambda + 2\mu$, which was the particular object of study in the first paper, is subject to relatively great changes of values, simply because it is the difference of two ratios, λ and -2μ, of which sometimes one and sometimes the other is the greater. Thus, the

measured volume changes of bore of the Iron Tubes I. to VII. (see Plates I. and II. of the previous paper) differ markedly from those of the Iron Tubes I′. to VIII′., and 3, 5, and 7. Nevertheless, the dilatations of Tube VII., as given in Table VIII. below, do not appreciably differ from those of other tubes.

It is this consideration which fully explains the apparently extraordinary behaviour of the steel tubes as described in the first paper (p. 537 and Plates III. and IV., *l.c.*). I have no measurement of the volume change of steel, so that it is not possible to calculate even the *tentative* values of ν, μ', and ν'. Probably δ is small compared with λ, and some idea of the character of the strain in steel might be got by assuming δ to be negligible. The results would not, however, differ essentially from those for iron. I have given in Table X. simply the undoubted results for the Steel Tubes, 3, 5, 7, VII.$_1$, VII.$_2$, VII.$_3$, and 9. There are three sets of columns of numbers. The first contains the ratios $\lambda + 2\mu$ calculated mainly from the data given in Part I.; the second contains the measured values of λ; and the third contains the calculated values of μ. Broadly speaking, they are very similar to the corresponding values for iron.

The history of Steel No. VII., as given in the Appendix to Part I., is very extraordinary. In its earliest condition distinguished as VII.$_1$ after the final boring which changed VI. into VII., it behaved, as regards bore dilatation, in a manner altogether peculiar. There was great positive dilatation up to Field 280, and negative dilatation in higher fields. Two months later, the law of the dilatation was found to be just reversed. This condition is distinguished as VII.$_2$. The tube was then annealed by slow cooling; and the bore dilatations were found to be greatly diminished in value, but otherwise to resemble roughly the corresponding quantities in the second condition. The contrasts between these three successive states of the same tube are shown in the numbers in Table X. The transverse dilatations μ for Steel VII.$_1$ and Steel VII.$_2$ are calculated on the assumption that the longitudinal dilatation of each is the same as that for VII.$_3$. A consideration of the three cases shows very plainly the effect of comparatively small changes in the values of the tangential dilatations. In none of them does 2μ differ greatly from $-\lambda$. Thus the ratio $\lambda + 2\mu$ is comparatively small in all. In VII.$_1$, λ has the preponderating influence; in VII.$_2$, 2μ preponderates. In VII.$_3$, 2μ still preponderates, but very slightly; so that in this case the bore dilatations are very insignificant compared to the linear dilatations.

§ 9. THE COILED IRON TUBES C I., C II. (TABLE V.)—These were formed of sheet-iron ·37 mm. in thickness. They were each about 10 inches long. The diameter of C I. was shorter than 1 inch by 4 per cent., and the diameter of C II. longer by about the same fraction.

No appreciable change of volume of the metal was obtained, but this simply means that it was too small to be measured. Under the most favourable conditions of experiment half a small division of the micrometer scale was about the limit of certainty. This would mean a volume change of about 2×10^{-6} cub. cm., giving a cubical dilatation of $\cdot 25 \times 10^{-6}$. I believe half of this value might have been detected. We may safely conclude that the cubical dilatation of this kind of iron, when placed in a magnetic field

of 500, does not exceed $\cdot 2 \times 10^{-6}$. In the calculation of the radial dilatations (ν') the cubical dilatation has been assumed to be zero. If, as is highly probable, the cubical dilatation is positive, the values for ν' will be slightly greater. There is, however, no pronounced difference between the values of μ' and ν', such as exists in the case of the coiled nickel tubes.

For the sake of easy comparison the $\delta v'$ curves for C I. and C II. are shown on Plate II., drawn to the same reference point and axes as the curves giving the volume changes for A V. They are the dotted curves, and are characterised by a maximum in Field 25, followed by a minimum in Field 120 or 130. They thus bear a certain resemblance to the $\delta v'$ curve for A I., in which, however, the early maximum is not strongly markèd.

The maximum elongation occurs in both cases about Field 60, a distinctly higher value than what is usually obtained with wires.

The broad distinction between the results for the bored and coiled tubes is the comparatively large distortion experienced by the latter in strong magnetic fields.

§ 10. General Conclusions.—It remains to give a general summary of the results, with special reference to the characteristics of the strains which accompany the magnetization of iron, nickel, and cobalt tubes. In a broad sense, the results for any one metal are much the same, whatever the dimensions of the tube may have happened to be. The changes in the values of the various dilatations, cubical or linear, as we pass from tube to tube of the same metal, are, for the most part, of secondary importance, and seem to belong to the same category of phenomena as the influence of form upon the susceptibility.

Of all the quantities measured, the changes of volume of bore and the changes of external volume of the plugged tube obey what, in certain cases, appear to be most capricious rules. But as soon as we calculate the strain coefficients, we see at once the reason for this apparent capriciousness. The strain coefficients themselves have, as already mentioned, very similar values in all the tubes of one metal; but as the longitudinal elongation (λ) usually differs in sign from the transverse elongation (μ), the ratio ($\lambda + 2\mu$) is arithmetically a difference of two numbers not very different in value. Sometimes the one term predominates, sometimes the other. Thus the A and B tubes Nos. V. and VI. are very similar as regards their dilatations (Table VII.), but there is at first sight a marked dissimilarity as regards their volume changes, δv and $\delta v'$ (Table VI.).

In my former paper I found great difficulty in interpreting the apparently capricious results obtained there. This difficulty now in great measure disappears, and the interest is transferred from the measured volume changes to the dilatations to which they lead.

As a general rule, the elongation in the direction of the magnetizing force is the most important. Mr Bidwell * has made us familiar with the laws of its variation in iron, nickel, and cobalt; and the values obtained by me are in full accord with his. The steady manner in which the maximum elongation point shifts into lower fields

* *Phil. Trans.*, Series A., vol. 179, 1888.

as the bore of the tube is widened is a new experimental fact; but it is one which might almost have been predicted from our previous knowledge of the magnetic properties of iron (see above, p. 465).

In the case of the narrow bored Nickel Tubes 1 and I. the radial elongation is greater numerically than the longitudinal dilatation, at least in the higher fields. This feature is slightly shown in Tube II., and is associated with a tangential elongation of the same sign as the longitudinal elongation. In these tubes, accordingly, the internal radius shortens, and the external radius lengthens, so that we are led to the conception of a cylindrical surface within the metal which experiences no displacement outwards or inwards.

A similar feature is characteristic of the narrower bored iron tubes in high fields, as a glance down the columns λ and ν in Table VII. will show.

Now, if we compare the measured quantity $\lambda + 2\mu'$ for a nickel tube of very narrow bore with the quantity δ obtained for the bar from which the tube was formed, we find that the former quantity is numerically much the greater. This shows that the removal of a comparatively small amount of material from the core of a nickel bar alters, in a remarkable degree, the character of the strain which accompanies powerful magnetization. In the bar, therefore, the molecules must be subject to considerable mechanical constraint. The mere formation of a second free surface of small extent in the heart of the metal produces, so to speak, a relief, whose effects extend appreciably to the outer surface. There is, indeed, more resemblance between the narrowest and the widest bored tubes as regards their condition of strain in a magnetic field than between the narrowest bored tube and the original bar.

On the other hand, if we compare the quantity $\lambda + 2\mu'$ for an iron tube of very narrow bore with the quantity δ, we find that the former is the smaller, or, at least, is never the larger. A comparison of the quantities $\delta v'$ and δV for A I. and B I. in Table VI. or in Plate II. indicates this clearly enough. That is to say, the removal of a small amount of material in the heart of the bar alters, to a comparatively slight extent, the displacement of the outer surface under magnetization. The alteration is, nevertheless, quite unmistakable, the formation of a second free surface in the heart of the metal distinctly modifying its behaviour. The tendency, in both iron and nickel, is for $\delta v'$ to be *algebraically less* that δV; but numerically the difference is very much less in iron than in nickel.

Another point of interest is the *form* of ellipsoid into which a small spherical element at either surface is changed; and here again the comparative simplicity of the results for nickel is noteworthy. In every case the shortest axis of the magnetic strain ellipsoid is in the direction of magnetization, and in the great majority of cases the longest is in the direction of the radius. Occasionally in the tubes of narrowest bore the third axis, that which corresponds to μ, the tangential elongation, is one of contraction, but generally it is, like the radial axis, one of elongation. Thus, in all cases of the bored tubes, a circular element in any plane perpendicular to the axis of the cylinder

becomes an ellipse drawn out radially. In the coiled tubes, on the other hand, this ellipse is drawn out more in the tangential than in the radial direction, a feature which also belongs to the (coiled) cobalt tube. A very similar, indeed almost identical, result is obtained with the iron tubes. The central section of the strain ellipsoid in a plane perpendicular to the axis of the cylinder has its greater axis parallel to the radius of the cylinder; or, in other words, μ is either a greater negative ratio or a smaller positive ratio than ν. The only exceptions are tubes A VI. and VII., B VI. and VII. and the coiled tube C II.

In general, then, we may draw the following conclusions regarding the behaviour of tubes of the magnetic metals, iron, nickel, and cobalt:—

1. *The strain is one in which there is comparatively little change of volume, but considerable change of form; in other words, the magnetic stresses, acting on the molecular groups, are mainly shearing stresses. This is particularly true of nickel.*

2. *Except in the case of very thin walls, a circular element in any transverse plane perpendicular to the axis of the cylinder becomes, when the cylinder is magnetized parallel to its axis, an ellipse with its major axis pointing towards the axis of the tube. When the walls are thin, the ellipse into which a small circular element is changed has its minor axis pointing towards the axis of the tube. The ellipse, in both cases, increases in excentricity as the distance from the axis diminishes.*

To obtain similar results with purely mechanical stresses acting on the surfaces of a tube, we should have to take the normal stress on the external surface greater than that on the internal surface for all but the tubes of widest bore; and for the wide bored tubes we should have to take the internal normal stress the greater. For nickel and cobalt in all fields, and for iron in high fields, these surface stresses would be tensions; but for iron in low fields they would be pressures. This comparison is merely intended to illustrate the character of the strain, for there can be no fundamental resemblance between the magneto-elastic problem discussed in this paper and the purely elastic problem hinted at. In the one case we are dealing with the effect of surface tractions upon a tube of elastic material; in the other, with the effect of magnetic body forces upon a tube of elastic material of high susceptibility.

One invariable characteristic of the strain ellipsoid in nickel and cobalt is that the minimum axis is parallel to the magnetizing force. This characteristic holds for iron when the magnetizing force is distinctly higher than that which corresponds to the change of sign of the longitudinal elongation. In other words, when the iron shows marked contraction in the direction of magnetization, its behaviour, as determined by the strain ellipsoid, is very similar to the behaviour of the other metals.

Then, again, when the magnetizing force is such as to produce distinct positive elongation in a direction parallel to its line of action, in this direction also lies the maximum axis of the strain ellipsoid. It is only during the transition condition, as the longitudinal dilatation changes sign, that the corresponding axis of the strain ellipsoid loses its maximum or minimum characteristic.

These broad results are deduced from the numbers contained chiefly in Tables II. and VII. To facilitate somewhat the comparison between iron and nickel I have drawn up the following table giving for the Tubes B V. (iron and nickel) the dilatations in chosen fields at the inner and outer surfaces, thus indicating the ellipsoidal form into which an originally spherical element is changed under the action of magnetic force. The chosen fields are the fields of maximum elongation and of zero elongation in the Iron Tube B V., and the greatest field reached in the experiments. The ratios λ, μ, ν are one millionfold the elongations of three mutually perpendicular unit lines, which when added to unity give the principal semi-axes of the ellipsoid into which the unit sphere is changed.

Principal Elongations in Iron and Nickel Tubes B V., expressed in Millionths (10^{-6}).

	Field = 190.		Field = 310.		Field = 500.	
	Iron.	Nickel.	Iron.	Nickel.	Iron.	Nickel.
λ	+ 2·1	− 16·2	0	− 21·8	− 2·18	− 24·8
μ	− 1·5	+ 4·2	− ·4	+ 6·7	+ ·8	+ 6·9
ν	− ·4	+ 12·0	+ ·6	+ 15·7	+ 1·57	+ 17·9
λ'	+ 2·1	− 16·2	0	− 21·8	− 2·18	− 24·8
μ'	− 1·25	+ 6·5	− ·1	+ 8·4	+ 1·01	+ 9·6
ν'	− ·65	+ 9·7	+ ·3	+ 13·4	+ 1·26	+ 15·2

We have no right to assume that these values hold for every point along the inner and outer surfaces of the tube. Being calculated from observed total changes of length and volume for the whole tube, they are, at best, only average values; and it is highly probable that they vary as we pass from points at the middle to points near the ends. Nevertheless, since much longer tubes give very similar results, these average values must be fair approximations to the true values at most parts of the tube.

It is also a fair presumption that elements in the heart of the metal suffer strains intermediate in character to those belonging to the surface elements.

Taking, then, the mean of the two strains in each case just given, let us calculate the stresses associated with these strains, assumed for the moment to be elastic.

Let P, Q, R be the principal stresses corresponding to the elongations λ, μ, ν. Then P is of the form $M\delta - N\lambda$; and the corresponding expressions for Q and R are obtained by cyclical permutation of λ, μ, ν.

Expressed in terms of Young's modulus E and the rigidity n, the stresses become

$$P = n\left(\frac{E - 2n}{3n - E}\,\delta + 2\lambda\right),\ Q = \text{etc.},\ R = \text{etc.}$$

We may take, for rough estimation, $E = 2 \times 10^{12}$ and $n = \cdot 8 \times 10^{12}$ in both metals,* giving the following values in kilogrammes per square centimetre for the stresses corresponding to the mean strains in the foregoing table, if we consider these strains to be purely elastic.

PRINCIPAL MEAN STRESSES IN IRON AND NICKEL TUBES B V. IN KGMS. PER SQ. CM.

	Field = 190.		Field = 310.		Field = 500.	
	Iron.	Nickel.	Iron.	Nickel.	Iron.	Nickel.
P	+1·8	−13	+·08	−18	−1·6	−20
Q	−1·0	+4	−·1	+6	+·8	+7
R	−·6	+9	+·3	+12	+1·2	+13

From these numbers we may calculate for each state the quantity

$$\tfrac{1}{2}(P\lambda + Q\mu + R\nu) = \tfrac{1}{2}n\ \{\delta^2 + 2(\lambda^2 + \mu^2 + \nu^2)\}$$

which measures the potential energy stored up in unit volume of an elastic substance strained to the extent and in the manner indicated by the ratios λ, μ, ν. We find, then, for the potential energy of strain in the three fields named, the quantities

5·3, 0·2, and 6·1 ergs in iron.
327 , 595 , „ 766 „ „ nickel.

These are calculated by taking into account only the *final* values. But there is a fundamental difference between the manner in which the magnetic force of, say, 310 carries the iron through its intermediate conditions of strain, and the manner in which an application of surface tractions effects the same succession of states. At every stage in the process, whether the strain coefficients be increasing or decreasing, the magnetizing force is producing a change which we must suppose to be always resisted by the elastic forces—that is to say, between the field corresponding to the maximum elongation in iron and the zero elongation, there is no real *assisting* of the forces which are producing the strain. We have, in short, no warrant in assuming a giving back of energy by the strained metal during this stage, which, in the purely elastic problem, answers to a condition of work done *by* the substance as it recovers. Since the elastic constants are not appreciably changed by magnetization, we may plausibly enough assume the work done during the second half of the positive elongation stage to be equal to the work done during the first half. Consequently, instead of 0·2 erg being the amount of work done against the elastic forces in the Field 310, we should take 10·6 + 0·2 (= 10·8) as in all probability the better value.

* Mr Mitchell, a student in the Physical Laboratory, Edinburgh University, determined for me the value of YOUNG's modulus by flexure experiments on the nickel sheets from which the coiled Tubes C_2 C_3 had been formed. The values found were respectively $2{\cdot}1 \times 10^{12}$ and $2{\cdot}5 \times 10^{12}$ in C.G.S. units.

Assuming then that the applied magnetic force strains the substance at every stage against the molecular forces which determine the elasticity of the substance, we find that the amounts of work done per unit volume in so straining the two metals in the three fields named are respectively :—

5·3, 10·8, and 16·7 ergs in iron.
327 , 595 , „ 766 „ „ nickel.

A similar calculation applied to the Iron and Nickel Tubes B II., which are much thicker in the walls than B V., gives for the amounts of work done per unit volume in straining the substance by the magnetic forces 200, 300, and 500, the quantities

6·5, 17, and 27·2 ergs in iron.
385 , 730, „ 1114·6 „ „ nickel.

From both these examples, so very different in the details of the strains, we find, by taking the ratios of the corresponding pairs, that 60 to 40 times as much work is done in straining nickel as is done in straining iron by means of magnetic forces whose values lie between 200 and 500.

Applying a similar calculation to the coiled Tubes C II., of which the iron tube is, however, somewhat thinner in the wall than the nickel tube, we find for the amounts of work stored up in unit volume in Fields 60, 180, 300, and 500,

12·1, 24·1, 32·3, 66 ergs in iron.
99·7, 596 , 818·4, 961 „ „ nickel.

These quantities show that in a field of 250 the work done in straining nickel against the elastic stresses is about 27 times the corresponding work done in straining the iron. Because of the different thicknesses of the tubes, this comparison has not the same claim to attention as the other two.

The assumptions underlying the calculations that have just been made are, of course, open to criticism.

The magnetic force acts directly on the molecular groupings, which break up to form new configurations. The mutual action between every contiguous pair of these new configurations is in all probability the effective cause of the strains produced. We know nothing definite regarding this mutual action, but it is not an altogether unreasonable hypothesis to suppose that the strains are produced against the elastic forces which bind the molecules together. It is, at all events, a fact worthy of note, that the less susceptible material is the one on which the greater amount of molecular work is done.

I have made no attempt to bring these experimental facts into line with the theories of magneto-striction as developed by HELMHOLTZ, KIRCHHOFF, J. J. THOMSON, and others. CANTONE's calculation* of KIRCHHOFF's constants from the results of experiment on iron and nickel ovoids can hardly be regarded as a test of the applicability of KIRCHHOFF's

* *Atti d. R. Accad. d. Lincei*, vi. 1890.

theory. Dr TAYLOR JONES has proved* that only a small part of the longitudinal contractions of magnetized nickel wire can be accounted for by means of KIRCHHOFF'S or THOMSON'S theory. NAGAOKA and JONES † have shown that KIRCHHOFF'S theory, when applied to an anchor ring uniformly magnetized, leads to the conclusion that the cubical dilatation should be three times the linear dilatation—a result not borne out by BIDWELL'S experiments. ‡

Under the influence of strong magnetic forces, iron, nickel, and cobalt are no doubt brought into an æolotropic state, very different from the state before the magnetic forces were applied. The molecules or molecular groups are thrown into new configurations, which, by their mutual action, give rise to the accompanying strains. To get some idea of the nature of these strains under conditions favourable for fairly accurate measurement has been the object of this paper. The significance of many of the facts described is by no means clear, and no recognised theory of magnetic stress seems able to elucidate them. The introduction of terms representing the rotation of molecules § adds greatly to the complexity of the equations, and it is difficult to see how these could be experimentally tested. It is not, however, so much the rotation of the individual molecule that we have to consider, as the resultant effect of new configurations of molecular groups.

APPENDIX—*September* 1898.—As this paper was passing through the press, I received from my former pupil and colleague, Professor NAGAOKA, of the Imperial University, Japan, an important and masterly discussion ‖ of the applicability of KIRCHHOFF'S theory of magneto-striction to the co-ordination of the inter-relations of magnetism and strain. In this paper, NAGAOKA and HONDA give measurements of the volume changes due to magnetization in iron and nickel, which are fairly concordant with the values given here. The nickel rod they used was much smaller in section than my nickel B tube, which may, perhaps, account for the greater values of the cubical dilatation obtained by them. They compare their results with my measurements of the changes of volume of *bore* as given in Part I.; but such a comparison cannot really be made. It is not merely, as they suggest, that their "measurements of the volumes were *external*," while mine "were made on the changes in the *internal* capacity of a nickel tube." As already pointed out (see above, p. 473), the boring out of a nickel bar completely changes its behaviour in a magnetic field, the outer surface becoming subject to a displacement much greater than what was found with the bar solid throughout. For the same reason their remark that a certain inconvenience inseparable from my earlier form of experiment "will disappear if the change of volume

* *Phil. Trans.*, Series A., vol. 189, pp. 189–200, 1895.

† *Phil. Mag.*, May 1896, p. 454.

‡ *Proc. Roy. Soc.*, 1890.

§ See DUHEM, *American Journal of Mathematics*, xvii. p. 117, 1895.

‖ "Researches on Magneto-striction." By Professor NAGAOKA and Mr HONDA. *Journal of the College of Science*, Imperial University, Japan, vol. ix. p. 353; also in *Phil. Mag.* for September 1898.

of the magnet itself be observed" has no real relevancy; for the change of volume of the magnet itself was not the subject of inquiry.

NAGAOKA and HONDA's experimental determination of the very slight effect of hydrostatic pressure on the magnetization is of high interest. It emphasises the view expressed above that the strain accompanying magnetization is mainly a shear. Whether we accept KIRCHHOFF's theory or not, we should expect to find small cubical dilatation under magnetization to be associated with small magnetic change under increased hydrostatic pressure. This result, consequently, can hardly be regarded as a verification of KIRCHHOFF's theory. A similar remark may be made in regard to other reciprocal relations; and before such a theory as KIRCHHOFF's can be accepted as established, there should be approximate *numerical* identity between theory and experiment. Mere *qualitative* agreement in a few particulars between theory and experiment may be largely a matter of chance, and cannot be put in the same category with one serious discrepancy. NAGAOKA and HONDA admit that "KIRCHHOFF's theory is a rough approximation," but conclude "that, excepting the theoretical deduction as to the effect of hydrostatic pressure on the magnetization of iron, there are no serious discrepancies between theory and experiment." Their high class experimental work, by bringing to light one serious discrepancy and other discrepancies of a less serious character, seems to me to demonstrate the insufficiency of KIRCHHOFF's theory. This, of itself, is important enough; but, in the present state of our knowledge, the new experimental facts discovered by NAGAOKA and HONDA are of much greater importance.

EXPLANATION OF SYMBOLS USED IN THE FOLLOWING TABLES.

V = volume of material of tube. v = ,, bore ,, v' = ,, bar of same length and breadth as tube.	All expressed in cubic centimetres.

δV, δv, $\delta v'$ = the measured changes of the volumes; unit, 10^{-6} cub. cm.

$\delta = \delta V/V$ = cubical dilatation. λ = elongation parallel to axis of tube. μ, μ' = tangential elongations at inner and outer surfaces. ν, ν' = radial ,, ,, ,,	All expressed in millionths, 10^{-6}.

The value of the magnetic field in every case is the value at the centre of the magnetizing coil and is expressed in C.G.S. magnetic units.

TABLE I.—Dimensions of the Various Tubes.

The Nickel Tubes (Bored).

Tube.	Length.	Diameter. External.	Diameter. Internal.	Volume of Bore.	Volume of Metal.	Tube.	Length.	Diameter. External.	Diameter. Internal.	Volume of Bore.	Volume of Metal.
B II.	20·2	2·77	·953	14·04	103·36	I.	47	4·2	·635	17·43	633·57
" III.	"	"	1·270	25·18	92·22	II.	"	"	·953	33·57	617·43
" IV.	"	"	1·588	38·80	78·60	III.	"	"	1·270	59·77	591·23
" V.	"	"	1·905	58	59·4	IV.	"	"	1·588	95·81	555·19
1	47	4·2	·635	17·88	633·12	V.	"	"	1·905	136·35	514·65
4	"	"	1·588	87·58	563·42	VI.	"	"	2·222	173·72	477·28
7	"	"	2·540	224·47	426·53	VII.	"	"	2·540	228·6	422·4

The Coiled Tubes.

Tube.	Length.	Diameter.	Thickness.	Volume of Bore.	Volume of Metal.
Nickel C I.	25	2·5	·027	117	5·85
" C II.	"	"	·050	113	9·78
" C III.	"	"	·105	102·8	20·01
Cobalt.	25·2	2·42	·038	112·4	7·6
Iron C I.	25·5	2·63	·037	127·2	8·3
" C II.	25·6	2·4	"	118·25	8·25

Iron Tubes A and B.

Tube.	Length.	Diameter. External.	Diameter. Internal.	Volume of Bore.	Volume of Metal.	Tube.	Length.	Diameter. External.	Diameter. Internal.	Volume of Bore.	Volume of Metal.
A I.	40·6	2·62	·635	13·68	205·12	B I.	20·3	2·62	·635	7·52	101·88
" II.	"	"	·953	27·7	191·1	" II.	"	"	·953	14·44	94·96
" III.	"	"	1·270	52·12	166·68	" III.	"	"	1·270	25·78	83·62
" IV.	"	"	1·588	79·18	139·62	" IV.	"	"	1·588	39·11	70·29
" V.	"	"	1·905	112·35	106·45	" V.	"	"	1·905	55·26	54·14
" VI.	"	"	2·222	152·48	66·32	" VI.	"	"	2·222	76·17	33·23
" VII.	"	"	2·420	184·63	34·17	" VII.	"	"	2·420	93·59	15·8

Iron and Steel Tubes (Original Set).

Iron.						Steel.					
Tube.	Length.	Diameter. External.	Diameter. Internal.	Volume of Bore.	Volume of Metal.	Tube.	Length.	Diameter. External.	Diameter. Internal.	Volume of Bore.	Volume of Metal.
3	46·2	3·84	1·270	67·7	470·1	3	45·7	3·84	1·270	61·93	467·17
5	"	"	1·905	127·4	410·4	5	"	"	1·905	127·95	401·15
7	"	"	2·540	223·4	314·4	7	"	"	2·540	227·45	301·65
VII.	"	"	2·696	252·08	285·72	VII.	"	"	2·696	243·51	285·59
9	"	"	3 19	343·7	194·1	9	"	"	3·19	343·3	185·8

Iron Tubes I'. to VIII'.

Tube.	Length.	Diameter. External.	Diameter. Internal.	Volume of Bore.	Volume of Metal.
I'.	46·2	3·84	·635	17·4	511·7
II'.	"	"	·953	31·96	497·14
III'.	"	"	1·270	59·6	469·5
IV'.	"	"	1·588	90 38	438·72
V'.	"	"	1·905	128 8	400 3
VI'.	"	"	2·222	173·13	355·97
VII'.	"	"	2·540	220·82	308·28
VIII'.	"	"	2·858	286 71	242·39

TABLE II.

Nickel Tubes B – II, III, IV, V.

Field.	Volume Changes × 10^6 c.c.			Dilatations × 10^6					
	δV.	δv.	$\delta v'$.	δ.	λ.	μ.	ν.	μ'.	ν'.
50	− ·5	0	+ 2·5	− ·00	− ·5	+ ·25	+ ·25	+ ·25	+ ·25
100	− 2·5	− 5	− 5·5	− ·02	− 3·5	+ 1·6	+ 1·9	+ 1·7	+ 1·8
150	− 5·4	− 55	− 57	− ·05	− 8·5	+ 2·3	+ 6·2	+ 4·0	+ 4·5
200	− 8·0	− 102	− 109·3	− ·08	− 12·4	+ 2·6	+ 9·7	+ 5·7	+ 6·6
250	− 7·8	− 140	− 150	− ·08	− 15·2	+ 2·6	+ 12·5	+ 7·0	+ 8·1
300	− 6·2	− 167	− 171·7	− ·06	− 16·9	+ 2·5	+ 14·4	+ 7·2	+ 9·7
400	− 2·8	− 192	− 197·8	− ·03	− 19·3	+ 2·8	+ 16·5	+ 8·8	+ 10·5
500	+ 3·3	− 210	− 212·7	+ ·03	− 21	+ 3	+ 18	+ 9·6	+ 11·4
50	− ·8	0	0	− ·01	− 1·2	+ ·6	+ ·6	+ ·6	+ ·6
100	− 2·7	− 51	− 59	− ·03	− 5·5	+ 1·8	+ 3·7	+ 2·5	+ 3
150	− 5·5	− 172	− 190	− ·06	− 10·2	+ 1·7	+ 8·5	+ 4·3	+ 5·9
200	− 7·8	− 278	− 294	− ·09	− 14·7	+ 1·9	+ 12·7	+ 6·1	+ 8·5
250	− 7·7	− 346	− 362	− ·09	− 17·4	+ 1·9	+ 15·4	+ 7·2	+ 10·1
300	− 6·2	− 387	− 403	− ·07	− 19·4	+ 2·0	+ 17·4	+ 8	+ 11·4
400	− 2·8	− 435	− 444	·03	− 21·7	+ 2·2	+ 19·5	+ 9	+ 12·7
500	+ 3·1	− 456	− 459	+ ·04	− 23·1	+ 2·5	+ 20·6	+ 9·6	+ 13·5
50	0	+ 1	− 2	− ·00	− 1·6	+ ·8	+ ·8	+ ·8	+ ·8
100	− ·5	− 80	− 80	− ·01	− 6·4	+ 2·2	+ 4·2	+ 2·9	+ 3·5
150	− 1	− 208	− 217	− ·01	− 11·4	+ 3·0	+ 8·4	+ 4·8	+ 6·6
200	− 1·3	− 303	− 321	− ·02	− 15·5	+ 3·8	+ 11·7	+ 6·4	+ 9·1
250	− 2	− 364	− 384	− ·03	− 18·2	+ 4·4	+ 13·8	+ 7·5	+ 10·7
300	− 2·7	− 401	− 416	− ·03	− 20	+ 4·7	+ 15·3	+ 8·2	+ 11·8
400	− 2·8	− 436	− 448	− ·04	− 22·2	+ 5·5	+ 16·7	+ 9·2	+ 13
500	+ 3·7	− 450	− 463	− ·05	− 23·8	+ 6·1	+ 17·7	+ 10	+ 13·8
50	0		− 80	− ·00	− 2	+ ·8	+ 1·2	+ ·9	+ 1·1
100	− ·7		− 185	− ·01	− 8·3	+ 2·6	+ 5·7	+ 3·4	+ 4·9
150	− 1		− 331	− ·02	− 13·3	+ 3·8	+ 9·5	+ 5·2	+ 8·1
200	− ·8		− 481	− ·01	− 17	+ 4·4	+ 12·6	+ 6·5	+ 10·5
250	− 1·2		− 527	− ·02	− 19·8	+ 5·4	+ 14·4	+ 7·7	+ 12·1
300	− 1·5		− 583	− ·03	− 21·7	+ 5·9	+ 15·8	+ 8·4	+ 13·3
400	− 2·5		− 615	− ·04	− 24	+ 6·8	+ 17·2	+ 9·4	+ 14·6
500	− 3·4		− 627	− ·06	− 24·8	+ 6·9	+ 17·9	+ 9·6	+ 15·2

TABLE III.

Nickel Tubes 1, 4, 7, VII.

Field.	λ.	μ.	ν.	μ'.	ν'.		δv.
50	− 1·6	+ ·8	+ ·8	+ ·8	+ ·8		− 1
100	− 6·8	+ 1·1	+ 5·7	+ 3·3	+ 3·5		− 82
150	− 11·9	− ·9	+ 12·8	+ 5·8	+ 6·1		− 245
200	− 15·5	− 3·0	+ 18·5	+ 7·5	+ 8·0	1	− 383
300	− 19·3	− 4·6	+ 23·9	+ 9·2	+ 10·1		− 509
400	− 21·7	− 6·1	+ 27·8	+ 10·4	+ 11·3		− 605
500	− 23·1	− 6·2	+ 29·3	+ 11·1	+ 12·0		− 634
50	− 3·2	+ 1·5	+ 1·7	+ 1·6	+ 1·6		− 26
100	− 9·8	+ 2·9	+ 6·9	+ 4·6	+ 5·2		− 358
150	− 14·6	+ 3·7	+ 10·9	+ 6·8	+ 7·8		− 639
200	− 17·5	+ 4·2	+ 13·3	+ 8·1	+ 9·4	4	− 794
300	− 20·8	+ 5·1	+ 15·7	+ 9·7	+ 11·1		− 924
400	− 22·4	+ 5·7	+ 16·7	+ 10·5	+ 11·9		− 967
500	− 23·2	+ 6·0	+ 17·2	+ 10·9	+ 12·3		− 975
50	− 3·8	+ 1·4	+ 2·4	+ 1·7	+ 2·1		− 210
100	− 10·4	+ 2·5	+ 7·9	+ 4·3	+ 6·1		− 1230
150	− 15·3	+ 3·3	+ 12·0	+ 6·2	+ 9·1		− 1920
200	− 18·2	+ 3·8	+ 14·4	+ 7·3	+ 10·9	7	− 2390
300	− 21·5	+ 4·7	+ 16·8	+ 8·6	+ 12·9		− 2730
400	− 22·8	+ 5·2	+ 17·6	+ 9·2	+ 13·6		− 2800
500	− 23·6	+ 5·4	+ 18·2	+ 9·6	+ 14·0		− 2870
50	− 3·6	+ 1·7	+ 1·9	+ 1·8	+ 1·8		− 45
100	− 9·6	+ 3·3	+ 6·3	+ 4·7	+ 4·9		− 710
150	− 14·5	+ 4·3	+ 10·2	+ 6·2	+ 8·3		− 1350
200	− 17·5	+ 4·9	+ 12·6	+ 7·4	+ 10·1	VII.	− 1750
300	− 20·8	+ 5·8	+ 15·0	+ 8·8	+ 12·0		− 2130
400	− 22·4	+ 6·3	+ 16·2	+ 9·5	+ 12·9		− 2285
500	− 23·5	+ 6·7	+ 16·8	+ 10·0	+ 13·5		− 2335

TABLE IV.

NICKEL TUBES, I.–VII.

Field.	λ.	μ.	ν.	μ′.	ν′.		δv.
50	− 3·6	+ 2·0	+ 1·6	+ 1·9	+ 1·7	I.	+ 8
150	− 14·5	+ 1·7	+ 12·8	+ 7·1	+ 7·4		− 195
300	− 20·8	− 3·8	+ 24·6	+ 10·0	+ 10·8		− 495
500	− 23·5	− 4·2	+ 27·2	+ 11·3	+ 12·2		− 555
50	− 3·6	+ 2·0	+ 1·6	+ 1·9	+ 1·7	II.	+ 15
150	− 14·5	+ 3·2	+ 11·3	+ 7·0	+ 7·5		− 270
300	− 20·8	+ ·9	+ 19·9	+ 9·9	+ 10·9		− 640
500	− 23·5	− ·9	+ 24·4	+ 11·1	+ 12·4		− 850
50	− 3·6	+ 2·2	+ 1·4	+ 1·8	+ 1·8	III.	+ 45
150	− 14·5	+ 3·6	+ 10·9	+ 6·9	+ 7·6		− 440
300	− 20·8	+ ·8	+ 20	+ 9·5	+ 11·3		− 1160
500	− 23·5	+ ·3	+ 23·2	+ 10·6	+ 12·9		− 1380
50	− 3·6	+ 2·2	+ 1·4	+ 1·9	+ 1·7	IV.	+ 75
150	− 14·5	+ 4·9	+ 9·6	+ 6·9	+ 7·6		− 460
300	− 20 8	+ 4·4	+ 16·4	+ 9·5	+ 11·3		− 1160
500	− 23·5	+ 4·4	+ 19·1	+ 10·7	+ 12·8		− 1400
50	− 3·6	+ 2·2	+ 1·4	+ 1·9	+ 1·7	V.	+ 110
150	− 14·5	+ 5·5	+ 9·0	+ 6·9	+ 7·6		− 480
300	− 20·8	+ 6·4	+ 14·4	+ 9·6	+ 11·2		− 1100
500	− 23·5	+ 7·1	+ 16·4	+ 10·8	+ 12·7		− 1280
50	− 3·6	+ 1·9	+ 1·7	+ 1·8	+ 1·8	VI.	+ 25
150	− 14·5	+ 4·0	+ 10·5	+ 6·4	+ 8·1		− 1150
300	− 20·8	+ 4·7	+ 16·1	+ 8·9	+ 11·9		− 2000
500	− 23·5	+ 5·5	+ 18·0	+ 10·1	+ 13·4		− 2190
50	− 3·6	+ 1·7	+ 1·9	+ 1·8	+ 1·8	VII.	− 45
150	− 14·5	+ 4·3	+ 10·2	+ 6·2	+ 8·3		− 1350
300	− 20·8	+ 5·8	+ 15·0	+ 8·8	+ 12·0		− 2130
500	− 23·5	+ 6·7	+ 16·8	+ 10	+ 13·5		− 2335

TABLE V.

Nickel C Tubes (Coiled).

Field.	$\delta v'$.			C I.			C II.			C III.		
	C I.	C II.	C III.	λ.	μ'.	ν'.	λ.	μ'.	ν'.	λ.	μ'.	ν'.
25	− 54	− 22	− 22	− 4	+ 1·8	+ 2·2	− 2	+ 1·0	+ 1·0	− 1·3	·6	·7
50	+ 72	+ 30	− 33	− 9·5	+ 5·0	+ 4·5	− 7·1	+ 3·7	+ 3·4	− 5·3	2·5	2·8
100	+ 430	+ 200	+ 46	− 18·2	+ 10·9	+ 7·4	− 15·3	+ 8·5	+ 6·8	− 13·6	7·0	6·6
150	+ 618	+ 345	+ 155	− 22·5	+ 13·8	+ 8·7	− 20·1	+ 11·5	+ 8·6	− 18·8	10·0	8·8
200	+ 718	+ 410	+ 233	− 25·4	+ 15·6	+ 9·8	− 22·9	+ 13·1	+ 9·8	− 22·3	12·1	10·2
300	+ 737	+ 483	+ 320	− 28·2	+ 17·1	+ 11·1	− 26	+ 15·0	+ 11·0	− 25·5	14·1	11·4
400	+ 735	+ 497	+ 349	− 29·3	+ 17·7	+ 11·6	− 27·2	+ 15·6	+ 11·6	− 27·5	15·2	12·3
500	+ 728	+ 498	+ 361	− 30	+ 18·0	+ 12·0	− 28·2	+ 16·1	+ 12·1	− 28·5	15·7	12·8

Cobalt Tube (Coiled).

Field.	$\delta v'$.	δV.	δ.	λ.	μ'.	ν'.
50	− 6	− ·1	− ·01	− ·2	+ ·1	+ ·1
100	− 26	− ·2	− ·03	− ·8	+ ·3	+ ·5
200	− 82	− ·9	− ·12	− 2·3	+ ·8	+ 1·4
300	− 158	− 1·3	− ·17	− 4·0	+ 1·4	+ 2·4
400	− 240	− 1·4	− ·18	− 5·8	+ 1·9	+ 3·7
500	− 320	− 1·4	− ·18	− 7·6	+ 2·5	+ 4·9

Iron C Tubes (Coiled).

Field.	$\delta v'$.		C I.			C II.		
	C I.	C II.	λ.	μ'.	ν'.	λ.	μ'.	ν'.
25	+ 15·2	+ 51	+ 1·9	− ·94	− ·96	+ 2·1	− ·87	− 1·23
50	− 15·3	+ 20	+ 2·21	− 1·17	− 1·04	+ 3·1	− 1·48	− 1·62
100	− 35	+ 0·8	+ 1·67	− ·98	− ·69	+ 2·17	− 1·08	− 1·09
150	− 36·2	+ 1	+ ·49	− ·40	− ·09	+ ·9	− ·45	− ·45
200	− 32·1	+ 8	− ·63	+ ·18	+ ·45	− ·3	+ ·18	+ ·12
300	− 20·7	+ 24	− 2·92	+ 1·38	+ 1·54	− 2·61	+ 1·39	+ 1·22
400	− 8·7	+ 42	− 4·83	+ 2·38	+ 2·45	− 4·78	+ 2·54	+ 2·24
500	+ 3·2	+ 62	− 6·1	+ 3·07	+ 3·03	− 5·9	+ 3·18	+ 2·72

TABLE VI.—Volume Changes in Iron.

Field.	A Tubes.				B Tubes.		
	δV.	δv.	$\delta v'$.		δV.	δv.	$\delta v'$.
50	+ 3	− 2·2	+ ·3	I.	+ 1	− ·1	+ 1
100	+ 15	− 26·7	− 11·5		+ 2·4	− ·5	+ 1·7
150	+ 28·6	− 36	− 9·5		+ 4·3	− 2·2	+ 2·9
200	+ 37·2	− 37·5	− 1·8		+ 8·1	− 6·8	+ 4·4
250	+ 45·2	− 37·9	+ 4·2		+ 13	− 14·2	+ 5·8
300	+ 50·5	− 38·2	+ 11		+ 18·3	− 17·9	+ 7·0
400	+ 62·7	− 39·3	+ 25		+ 27·8	− 22·8	+ 10·0
500	+ 75·2	− 40·8	+ 41·7		+ 37·8	− 26·7	+ 14·5
50	+ 5·4	− 10	− 5	II.	+ ·7	− ·8	+ ·3
100	+ 14·8	− 53·5	− 35·5		+ 3·2	− 2·4	+ ·5
150	+ 29	− 63	− 30·5		+ 7·4	− 7·8	− ·7
200	+ 38·4	− 64·1	− 23·8		+ 11·7	− 16·3	− 5·1
250	+ 46·3	− 65·1	− 18·8		+ 18·3	− 32·3	− 14·2
300	+ 52·2	− 66·1	− 15·8		+ 23·7	− 41	− 17·3
400	+ 62·3	− 68·3	− 9·5		+ 32·6	− 43·5	− 12·3
500	+ 73·1	− 69	− ·8		+ 37·5	− 47	− 9·5
50	+ 4·3	− 35	− 30	III.	+ 1	− 1·2	− ·7
100	+ 16·5	− 115	− 95		+ 3·1	− 5	− 2·5
150	+ 29	− 112	− 81		+ 7	− 15	− 9·4
200	+ 35·2	− 113	− 74·8		+ 13·2	− 41	− 28·4
250	+ 40	− 113	− 69·8		+ 19·9	− 54	− 32·5
300	+ 43	− 118	− 68		+ 25·6	− 51·3	− 23·4
400	+ 49·2	− 114	− 64		+ 36·2	− 52·2	− 12·0
500	+ 53·2	− 115	− 57·3		+ 43·3	− 52·2	− 6·7
50	+ 4	− 52	− 50	IV.	+ ·5	− 3·5	− 8·5
100	+ 12	− 125	− 111		+ 1·5	− 12·8	− 13
150	+ 21	− 122	− 101		+ 4·3	− 34	− 34
200	+ 26·2	− 122·5	− 96		+ 8·1	− 59·2	− 56·8
250	+ 29·7	− 122·5	− 91		+ 11·8	− 55	− 48
300	+ 33·5	− 123	− 88		+ 13·8	− 51·7	− 42·7
400	+ 41·7	− 124	− 83		+ 16·2	− 48	− 38·5
500	+ 51·2	− 125	− 80		+ 17·3	− 48	− 35·2
50	+ 3·6	− 100	− 100	V.	+ ·7	− 7	− 5·5
100	+ 11·5	− 109	− 93		+ 1·8	− 25	− 22
150	+ 17·8	− 100	− 81		+ 4·1	− 51·5	− 46·8
200	+ 22·2	− 97	− 78		+ 7·5	− 49·7	− 42·3
250	+ 24·7	− 95	− 67		+ 9·1	− 43·5	− 34
300	+ 26	− 94	− 67		+ 9·7	− 40·8	− 29·4
400	+ 32	− 91	− 62		+ 9·9	− 36·3	− 23·8
500	+ 39·7	− 88	− 58		+ 10·2	− 32·2	− 18·2
50	+ 4·8	− 38	− 33	VI.	+ 1	− 15·5	− 15
100	+ 8·9	+ 32	+ 46		+ 2·7	− 29·6	− 25
150	+ 12·6	+ 55·5	+ 71		+ 5·2	+ 4	+ 16
200	+ 16·2	+ 63·5	+ 80		+ 7·2	+ 22·5	+ 37·9
250	+ 20	+ 68	+ 86		+ 7·7	+ 32·3	+ 48
300	+ 22	+ 71·5	+ 91		+ 7·8	+ 39·2	+ 53·8
400	+ 26	+ 78·1	+ 98		+ 7·2	+ 47·5	+ 59·7
500	+ 28·7	+ 84·3	+ 128		+ 7·3	+ 53·5	+ 61·8
25	+ 2	− 18	Not observed.	VII.	+ ·2	− 8	− 7·7
50	+ 4	+ 42			+ ·5	− 13·2	− 10·2
100	+ 5·5	+ 97			+ ·9	+ 12·2	+ 17·5
150	+ 6·2	+ 107			+ 1·8	+ 20·4	+ 28·3
200	+ 6·5	+ 113			+ 2·7	+ 22·5	+ 31·5
300	+ 5	+ 118			+ 3·5	+ 23	+ 33
400	+ 3·2	+ 122			+ 3·2	+ 28·5	+ 33·5
500	+ 1·5	+ 132			+ 2·3	+ 26	+ 35·7

TABLE VII.—Dilatations in Iron.

Field.	A Tubes.							B Tubes.					
	δ.	λ.	μ.	ν.	μ'.	ν'.		δ.	λ.	μ.	ν.	μ'.	ν'.
50	+ ·02	+ ·6	− ·38	− ·20	− ·3	− ·28	I.	+ ·01	+ ·15	− ·08	− ·06	− ·07	− ·07
100	+ ·08	+ 1·8	− 1·88	+ ·16	− ·93	− ·8		+ ·02	+ ·48	− ·28	− ·18	− ·22	− ·23
150	+ ·14	+ 2·65	− 2·64	+ ·13	− 1·35	− 1·16		+ ·04	+ ·91	− ·6	− ·27	− ·44	− ·43
200	+ ·19	+ 2·5	− 2·62	+ ·31	− 1·25	− 1·06		+ ·08	+ 1·46	− 1·18	− ·20	− ·71	− ·67
250	+ ·23	+ 1·96	− 2·37	+ ·64	− ·96	− ·77		+ ·13	+ 2·08	− 1·99	+ ·04	− 1·02	− ·98
300	+ ·25	+ 1·34	− 2·07	+ ·98	− ·63	− ·44		+ ·18	+ 2·5	− 2·44	+ ·12	− 1·22	− 1·10
400	+ ·31	− ·2	− 1·34	+ 1·85	+ ·17	+ ·36		+ ·27	+ 2·79	− 2·91	+ ·39	− 1·35	− 1·17
500	+ ·38	− 1·6	− ·69	+ 2·71	+ ·90	+ 1·08		+ ·37	+ 2·42	− 2·99	+ ·94	− 1·15	− ·9
50	+ ·03	+ ·6	− ·48	− ·09	− ·31	− ·26	II.	+ ·01	+ ·17	− ·12	− ·04	− ·08	− ·08
100	+ ·08	+ 2·2	− 2·07	− ·05	− 1·18	− ·94		+ ·03	+ ·5	− ·34	− ·13	− ·25	− ·22
150	+ ·15	+ 2·72	− 2·5	− ·07	− 1·43	− 1·14		+ ·08	+ 1·0	− ·77	− ·15	− ·51	− ·41
200	+ ·20	+ 2·43	− 2·37	+ ·14	− 1·27	− ·96		+ ·12	+ 1·68	− 1·37	− ·14	− ·84	− ·67
250	+ ·24	+ 1·82	− 2·09	+ ·51	− ·95	− ·63		+ ·19	+ 2·28	− 2·26	+ ·17	− 1·21	− ·88
300	+ ·27	+ 1·1	− 1·74	+ ·91	− ·58	− ·28		+ ·25	+ 2·54	− 2·69	+ ·40	− 1·35	− ·94
400	+ ·33	− ·32	− 1·08	+ 1·78	+ ·15	+ ·50		+ ·34	+ 2·24	− 2·66	+ ·76	− 1·18	− ·72
500	+ ·38	− 1·63	− ·43	+ 2·44	+ ·8	+ 1·21		+ ·39	+ 1·3	− 2·28	+ 1·37	− ·70	− ·21
50	+ ·03	+ ·9	− ·79	− ·08	− ·52	− ·35	III.	+ ·01	+ ·24	− ·14	− ·09	− ·13	− ·10
100	+ ·10	+ 2·48	− 2·35	− ·08	− 1·46	− ·92		+ ·04	+ ·51	·35	− ·12	− ·27	− ·20
150	+ ·17	+ 2·75	− 2·45	− ·13	− 1·56	− 1·02		+ ·08	+ ·96	− ·77	− ·11	− ·53	− ·35
200	+ ·21	+ 2·42	− 2·29	+ ·08	− 1·38	− ·83		+ ·16	+ 1·45	− 1·52	+ ·28	− ·86	− ·43
250	+ ·24	+ 1·8	− 1·98	+ ·42	− 1·05	− ·51		+ ·24	+ 2·28	− 2·19	+ ·15	− 1·29	− ·75
300	+ ·26	+ 1·05	− 1·61	+ ·82	− ·67	− ·12		+ ·30	+ 2·17	− 2·08	+ ·21	− 1·20	− ·67
400	+ ·30	− ·4	− ·90	+ 1·60	+ ·06	+ ·64		+ ·43	+ 1·27	− 1·65	+ ·81	− ·69	− ·15
500	+ ·35	− 1·86	− ·17	+ 2·38	+ ·80	+ 1·41		+ ·52	− ·1	− ·96	+ 1·58	+ ·02	+ ·60
50	+ ·03	+ ·95	− ·81	− ·11	− ·59	− ·33	IV.	+ ·01	+ ·1	− ·10	+ ·1	− ·07	− ·02
100	+ ·09	+ 2·55	− 2·07	− ·39	− 1·58	− ·98		+ ·02	+ ·43	− ·38	− ·03	− ·28	− ·13
150	+ ·15	+ 2·58	− 2·06	− ·37	− 1·52	− ·91		+ ·06	+ 1·0	− ·94	·00	− ·66	− 28
200	+ ·19	+ 2·08	− 1·82	− ·07	− 1·26	− ·63		+ ·12	+ 2·0	− 1·76	− ·12	− 1·26	− ·62
250	+ ·21	+ 1·4	− 1·48	+ ·29	− ·9	− ·29		+ ·16	+ 2·0	− 1·71	− ·13	− 1·22	− ·62
300	+ ·24	+ 7	− 1·13	+ ·67	− ·54	+ ·08		+ ·20	+ 1·5	− 1·41	+ ·11	− ·95	− ·35
400	+ ·30	− ·7	− ·44	+ 1·44	+ ·17	+ ·88		+ ·23	+ ·45	− ·84	+ ·62	− ·45	+ ·23
500	+ ·37	− 1·97	+ ·20	+ 2·14	+ ·80	+ 1·54		+ ·25	− ·7	− ·26	+ 1·21	+ ·19	+ ·76
50	+ ·03	+ 1·15	− 1·02	− ·1	− ·81	− ·81	V.	+ ·01	+ ·1	− ·12	+ ·03	− ·08	− ·01
100	+ ·11	+ 2·16	− 1·57	− ·48	− 1·30	− ·75		+ ·03	+ ·7	− ·58	− ·09	− ·46	− ·21
150	+ ·17	+ 1·95	− 1·42	− ·36	− 1·16	− ·62		+ ·08	+ 1·5	− 1·22	− ·20	− ·97	− ·45
200	+ ·21	+ 1·37	− 1·12	− ·04	− ·85	− ·31		+ ·14	+ 2·1	− 1·50	− ·46	− 1·25	− ·71
250	+ ·23	+ ·61	− ·73	+ ·35	− ·46	+ ·08		+ ·17	+ 1·04	− ·93	+ ·05	− ·68	− ·19
300	+ ·24	− ·12	− ·36	+ ·72	− ·10	+ ·46		+ ·18	+ ·13	− ·44	+ ·47	− ·20	+ ·25
400	+ ·30	− 1·66	+ ·43	+ 1·53	+ ·69	+ 1·27		+ ·18	− 1·15	+ ·25	+ 1·03	+ ·47	+ ·76
500	+ ·37	− 2·7	+ ·96	+ 2·11	+ 1·28	+ 1·84		+ ·19	− 2·18	+ ·8	+ 1·57	+ 1·01	+ 1·26
50	+ ·07	+ 1·8	− 1·08	− ·7	− ·98	− ·75	VI.	+ ·03	+ ·3	− ·25	− ·02	− ·22	− ·05
100	+ ·13	+ 2·35	− 1·07	− 1·15	− 1·07	− 1·15		+ ·08	+ 1·7	− 1·05	− ·57	− ·97	− ·65
150	+ ·19	+ 1·74	− ·69	− ·86	− ·70	− ·85		+ ·15	+ 2·3	− 1·12	− 1·08	− 1·08	− 1·07
200	+ ·24	+ 1·07	− ·38	− ·50	− ·35	− ·48		+ ·22	+ 2·0	− ·85	− ·93	− ·83	− ·95
250	+ ·30	+ ·22	+ ·12	− ·04	+ ·09	− ·01		+ ·23	+ 1·3	− ·44	− ·59	− ·43	− ·64
300	+ ·33	− ·61	+ ·54	+ ·40	+ ·52	+ ·42		+ ·24	+ ·4	+ ·06	− ·22	+ ·1	− ·26
400	+ ·39	− 2·05	+ 1·28	+ 1·16	+ 1·25	+ 1·19		+ ·22	·96	+ ·79	+ ·39	+ ·76	+ ·42
500	+ ·43	− 3·1	+ 1·83	+ 1·70	+ 1·84	+ 1·69		+ ·22	− 2·1	+ 1·40	+ ·92	+ 1·34	+ ·98
25	+ ·06	+ 1·2	− ·65	− ·49	− ·64	− ·50	VII.	+ ·01	+ ·15	− ·11	− ·08	− ·13	− ·01
50	+ ·12	+ 2·2	− ·99	− 1·09	− ·99	− 1·09		+ ·03	+ 1·15	− ·64	− ·48	− ·63	− ·49
100	+ ·16	+ 2·4	− ·92	− 1·32	− ·99	− 1·25		+ ·06	+ 2·0	− ·94	− 1·00	− ·92	− 1·02
150	+ ·18	+ 1·7	− ·56	− ·96	− ·59	− ·93		+ ·12	+ 1·4	− ·59	− ·69	− ·57	− ·71
200	+ ·19	+ ·9	− ·15	− ·56	− ·18	− ·53		+ ·17	+ ·6	− ·10	− ·27	− ·14	− ·29
300	+ ·15	− ·55	+ ·60	+ ·1	+ ·56	+ ·14		+ ·22	− 1·42	+ ·84	+ ·80	+ ·86	+ ·78
400	+ ·09	− 1·63	+ 1·15	+ ·57	+ 1·11	+ ·61		+ ·20	− 2·96	+ 1·61	+ 1·65	+ 1·63	+ 1·53
500	+ ·04	− 2·6	+ 1·66	+ ·98	+ 1·61	+ 1·03		+ ·15	− 4·48	+ 2·38	+ 2·25	+ 2·40	+ 2·23

TABLE VIII.

Iron Tubes 3, 5, 7, VII., 9.

Field.	δ.	λ.	μ.	ν.	μ'.	ν'.		δv.
50	+ ·03	+ ·40	− ·21	− ·16	− ·19	− ·18		− ·9
100	+ ·09	+ 1·48	− 1·13	− ·26	− ·76	− ·63		− 52·9
150	+ ·17	+ 2·75	− 2·42	− ·16	− 1·44	− 1·14		− 141
200	+ ·22	+ 3·07	− 2·70	− ·15	− 1·59	− 1·26	3	− 158
300	+ ·28	+ 1·92	− 2·29	+ ·65	− 1·01	− ·63		− 181
400	+ ·33	+ ·50	− 1·90	+ 1·73	− ·32	+ ·11		− 223
500	+ ·43	− ·96	− 1·37	+ 2·76	+ ·43	+ ·96		− 249
50	+ ·03	+ ·17	− ·21	+ ·07	− ·11	− ·03		− 31
100	+ ·09	+ 1·05	− 1·27	+ ·31	− ·67	− ·29		− 187
150	+ ·17	+ 1·90	− 1·79	+ ·06	− 1·09	− ·64		− 213
200	+ ·22	+ 1·94	− 1·68	− ·04	− 1·06	− ·66	5	− 182
300	+ ·28	+ ·90	− 1·19	+ ·57	− ·52	− ·10		− 187
400	+ ·33	− ·53	− ·54	+ 1·40	+ ·20	+ ·66		− 207
500	+ ·43	− 1·84	+ ·03	+ 1·94	+ ·87	+ 1·40		− 226
50	+ ·03	+ ·38	− ·33	− ·02	− ·24	− ·11		− 60
100	+ ·09	+ 1·60	− 1·16	− ·35	− ·92	− ·59		− 159
150	+ ·17	+ 2·00	− 1·23	− ·6	− 1·05	− ·78		− 102
200	+ ·22	+ 1·60	− 1·00	− ·38	− ·82	− ·56	7	− 89
300	+ ·28	+ ·15	− ·24	+ ·37	− ·07	+ ·20		− 74
400	+ ·33	− 1·33	+ ·51	+ 1·15	+ ·70	+ ·96		− 69
500	+ ·43	− 2·71	+ 1·19	+ 1·95	+ 1·41	+ 1·73		− 75
50	+ ·03	+ ·58	− ·38	− ·17	− ·33	− ·22		− 46
100	+ ·09	+ 2·52	− 1·21	− 1·22	− 1·21	− 1·29		+ 28
150	+ ·17	+ 2·98	− 1·16	− 1·65	− 1·30	− 1·51		+ 166
200	+ ·22	+ 2·50	− ·83	− 1·45	− 1·00	− 1·28	VII.	+ 212
300	+ ·28	+ 1·27	− ·16	− ·83	− ·34	− ·65		+ 242
400	+ ·33	− ·18	+ ·59	− ·08	+ ·41	+ ·10		+ 252
500	+ ·43	− 1·61	+ 1·31	+ ·73	+ 1·16	+ ·88		+ 254
50	+ ·03	+ 1·1	− ·73	− ·34	− ·66	− ·41		− 121
100	+ ·09	+ 2·7	− 1·45	− 1·16	− 1·40	− 1·21		− 69
150	+ ·17	+ 2·45	− 1·15	− 1·13	− 1·14	− 1·14		+ 55
200	+ ·22	+ 1·77	− ·76	− ·79	− ·76	− ·79	9	+ 90
300	+ ·28	− ·03	+ ·18	+ ·13	+ ·17	+ ·14		+ 110
400	+ ·33	− 1·90	+ 1·13	+ 1·10	+ 1·12	+ 1·11		+ 121
500	+ ·43	− 3·55	+ 1·94	+ 2·04	+ 1·96	+ 2·02		+ 110

TABLE IX.

Iron Tubes, I'. to VIII'.

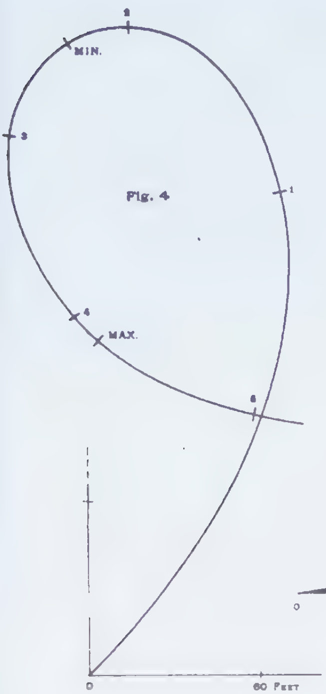
Field.	δ.	λ.	μ.	ν.	μ'.	ν'.		δv.
50	+ ·03	+ ·33	− ·01	− ·29	− 15	− ·15	I'.	+ 5·8
100	+ ·09	+ ·80	+ ·09	− ·80	− ·39	− ·32		+ 16·8
150	+ ·17	+ 1·80	− ·87	− ·76	− ·82	− ·81		+ 1·0
200	+ ·22	+ 2·25	− 1·43	− ·60	− 1·03	− 1·00		− 10·5
300	+ ·28	+ 2·00	− 1·65	− ·07	− ·89	− ·83		− 22·5
500	+ ·43	+ ·20	− 1·11	+ 1·33	+ ·08	+ ·15		− 35·1
50	+ ·03	+ 48	− ·24	− ·21	− ·23	− ·22	II'.	− ·2
100	+ ·09	+ 1·10	− ·66	− ·35	− ·51	− ·50		− 7
150	+ ·17	+ 2·00	− 1·94	+ ·11	− ·98	− ·85		− 60
200	+ ·22	+ 2·48	− 2·41	+ ·15	− 1·22	− 1·04		− 75
300	+ ·28	+ 1·85	− 2·27	+ ·70	− ·88	− ·69		− 86
500	+ ·43	− ·50	− 1·36	+ 2·29	− ·36	+ ·57		− 103
50	+ ·03	+ ·42	− ·25	− ·14	− ·20	− ·19	III'.	− 4
100	+ ·09	+ 1·30	− ·91	− ·30	− ·64	− ·57		− 30·4
150	+ ·17	+ 2·48	− 2·28	− ·03	− 1·28	− 1·03		− 124
200	+ ·22	+ 2·80	− 2·54	− ·04	− 1·43	− 1·15		− 136
300	+ ·28	+ 2·00	− 2·32	+ ·60	− 1·02	− ·7		− 157
500	+ ·43	− ·60	− 1·30	+ 2·33	+ ·31	+ ·72		− 190
50	+ ·03	+ ·56	− ·31	− ·22	− ·27	− ·26	IV'.	− 5·5
100	+ ·09	+ 1·43	− ·95	− ·39	− ·72	− ·62		− 42·2
150	+ ·17	+ 2·80	− 2·08	− ·55	− 1·45	− 1·18		− 124
200	+ ·22	+ 2·93	− 2·30	− ·41	− 1·52	− 1·19		− 151
300	+ ·28	+ 1·95	− 1·99	+ ·32	− 1·04	− ·63		− 183
500	+ ·43	− ·74	− ·87	+ 2·04	+ ·34	+ ·83		− 224
50	+ ·03	+ ·40	− ·28	− ·09	− ·21	− ·16	V'.	− 2
100	+ ·09	+ 1·64	− 1·28	− ·27	− ·81	− ·74		− 117
150	+ ·17	+ 2·70	− 2·26	− ·27	− 1·51	− 1·02		− 233
200	+ ·22	+ 2·64	− 2·27	− ·18	− 1·47	− ·95		− 244
300	+ ·28	+ 1·47	− 1·78	+ ·59	− ·89	− ·30		− 270
500	+ ·43	− 1·13	− ·56	+ 2·12	+ ·45	+ 1·11		− 295
50	+ ·03	+ ·45	− ·28	− ·14	− ·23	− ·19	VI'.	− 17
100	+ ·09	+ 2·10	− 1·41	− ·16	− 1·19	− ·82		− 125
150	+ ·17	+ 2·94	− 1·99	− ·78	− 1·59	− 1·18		− 180
200	+ ·22	+ 2·60	− 1·86	− ·52	− 1·41	− ·97		− 192
300	+ ·28	+ 1·38	− 1·32	+ ·22	− ·8	− ·30		− 218
500	+ ·43	− 1·20	− ·12	+ 1·75	+ ·51	+ 1·12		− 247
50	+ ·03	+ ·55	− ·44	− ·08	− ·38	− ·14	VII'.	− 72
100	+ ·09	+ 2·07	− 1·64	− ·34	− 1·26	− ·72		− 268
150	+ ·17	+ 2·47	− 1·81	− ·49	− 1·43	− ·87		− 254
200	+ ·22	+ 2·11	− 1·64	− ·25	− 1·24	− ·65		− 256
300	+ ·28	+ ·79	− ·98	+ ·47	− ·56	+ ·05		− 255
500	+ ·43	− 1·89	+ ·37	+ 1·95	+ ·83	+ 1·49		− 254
50	+ ·03	+ ·55	− ·49	− ·03	− ·39	− ·13	VIII'.	− 120
100	+ ·09	+ 2·25	− 1·45	− ·71	− 1·28	− ·88		− 185
150	+ ·17	+ 2·10	− 1·30	− ·63	− 1·15	− ·78		− 142
200	+ ·22	+ 1·48	− ·98	− ·28	− ·82	− ·44		− 139
300	+ ·28	·00	− ·23	+ ·51	− ·06	+ ·34		− 130
500	+ ·43	− 2·80	+ 1·10	+ 2·04	+ 1·39	+ 1·84		− 120

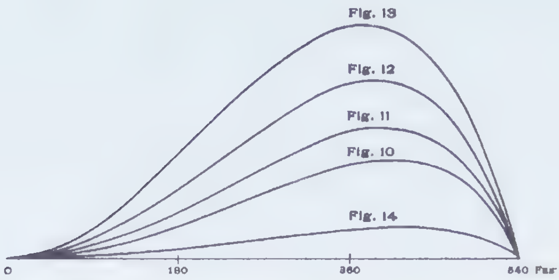
TABLE X.

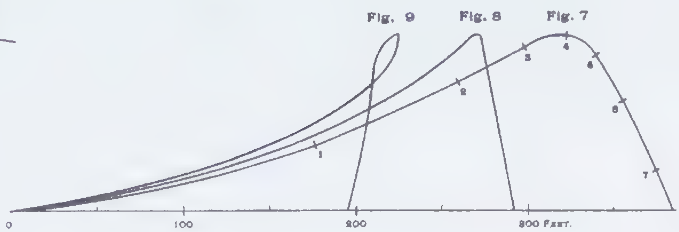
Steel Tubes.

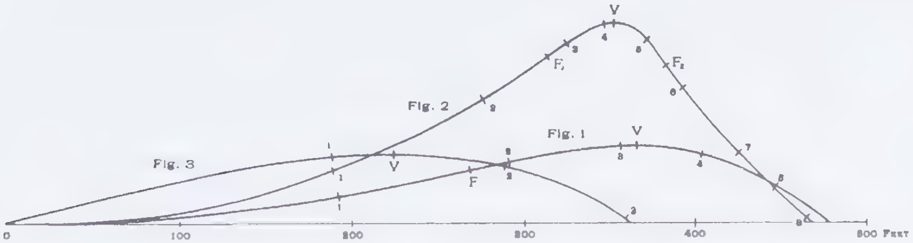
Field.	3.	5.	7.	VII_1.	VII_2.	VII_3.	9.
				Bore Dilatations, $(\lambda + 2\mu)10^6$.			
50	+ ·13	+ ·05	− ·19	+ ·05	− ·29	− ·10	− ·45
100	+ ·38	+ ·20	− ·30	+ ·25	− ·59	− ·02	+ ·08
150	+ ·16	+ ·68	+ ·21	+ ·40	− ·4	+ ·07	+ ·56
200	+ ·33	+ 1·25	+ ·52	+ ·31	− ·19	+ ·10	+ ·70
300	+ ·68	+ 2·14	+ ·76	− ·06	+ ·04	+ ·11	+ ·86
400	+ ·53	+ 2·64	+ ·96	− ·46	+ ·16	+ ·08	+ ·93
500	+ ·39	+ 2·89	+ 1·05	− ·82	+ ·24	+ ·04	+ ·98
				Longitudinal Dilatations, $\lambda\ 10^6$.			
50	+ ·25	+ ·15	+ ·51	Probably same as for VII_3.	Probably same as for VII_3.	+ ·68	+ ·95
100	+ ·68	+ ·85	+ 1·23			+ 2·20	+ 1·80
150	+ 1·40	+ 1·75	+ 1·47			+ 2·00	+ 1·50
200	+ 1·51	+ 1·80	+ ·98			+ 1·40	+ ·75
300	+ ·40	+ ·60	− ·45			− ·31	− ·90
400	− 1·00	− ·90	− 1·98			− 2·00	− 2·42
500	− 2·40	− 2·40	− 3·30			− 3·52	− 3·8
				Tangential Dilatations, $\mu\ 10^6$.			
50	− ·06	− ·05	− ·30	− ·32	− ·49	− ·39	− ·70
100	− ·15	− ·33	− ·77	− ·98	− 1·40	− 1·11	− ·86
150	− ·62	− ·54	− ·63	− ·80	− 1·20	− ·97	− ·47
200	− ·59	− ·28	− ·23	− ·55	− ·61	− ·65	− ·03
300	+ ·14	+ ·77	+ ·61	+ ·13	+ ·18	+ ·21	+ ·88
400	+ ·77	+ 1·77	+ 1·47	+ ·77	+ 1·08	+ 1·04	+ 1·68
500	+ 1·40	+ 2·65	+ 2·18	+ 1·35	+ 1·88	+ 1·77	+ 2·39

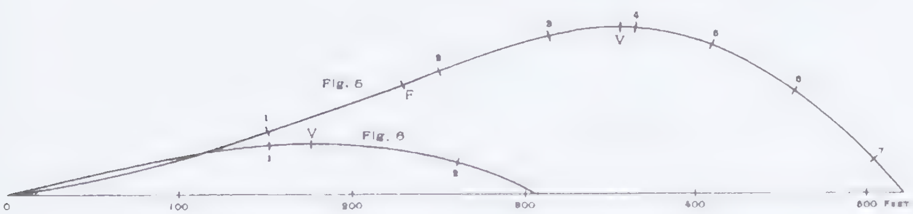
PROF. TAIT ON THE PATH OF A ROTATING SPHERICAL PROJECTILE.











XVI.—*On the Path of a Rotating Spherical Projectile.* II. By Professor TAIT. (With a Plate.)

(Read 6th and 20th January 1896.)

The first instalment of this paper was devoted in great part to the general subject involved in its title, but many of the illustrations were derived from the special case of the flight of a golf-ball. Since it was read I have endeavoured, alike by observation and by experiment, to improve my numerical data for this interesting application, particularly as regards the important question of the coefficient of resistance of the air. As will be seen, I now find a value intermediate to those derived (by taking average estimates of the mass and diameter of a golf-ball) from the results of ROBINS and of BASHFORTH. This has been obtained indirectly by means of a considerable improvement in the apparatus by which I had attempted to measure the initial speed of a golf-ball. I have, still, little doubt that the speed may, *occasionally*, amount to the 300, or perhaps even the 350, foot-seconds which I assumed provisionally in my former paper:—but even the first of these is a somewhat extravagant estimate; and I am now of opinion that, even with very good driving, an initial speed of about 240 is not often an underestimate, at least in careful play. From this, and the fact that six seconds at least are required for a long carry (say 180 yards), I reckon the "terminal velocity" at about 108, giving $v^2/360$ as the resistance-acceleration.

I hope to recur to this question towards the end of the present paper:—but I should repeat that I naturally preferred the comparatively recent determination to the much older one, and that in formerly assuming a resistance even greater than that which BASHFORTH's formula assigns, I was to some extent influenced by the consideration of the important effects of roughening or hammering a golf-ball. For I fancied that this might increase the direct resistance, as well as the effects due to rotation, by the better grip of the air which it gives to the ball. [See last sentence of § 11. Of course the assumption of increased coefficient of resistance *required* a corresponding increase of the estimate of initial speed.] The time of describing 180 yards *horizontally*, *i.e.*, when gravity is not supposed to act, if the initial speed is 240 and the "terminal velocity" 108, is about $5^s{\cdot}2$; and this has to be increased by at least 1^s, if we allow for the curvature of the path and the effect of gravity. I have employed this improved value of the coefficient of resistance in all the calculations which have been made since I obtained it. But various considerations have led me to the conclusion that the resistance, towards the end of the path, may be somewhat underrated because of the assumption that it is, throughout, proportional to the square of the speed. This point, also, will be referred to later, as I wish to make at once all the necessary comments and improvements on the part already published.

Though the present communication is thus specially devoted to some curious phenomena observed in the game of golf, it contains a great deal which has more extended application:—to which its results can easily be adapted by mere numerical alterations in the data. Therefore I venture to consider its subject as one suitable for discussion before a scientific Society.

In my short sketch of the history of the problem I failed to notice either of two comparatively recent papers whose contents are at least somewhat closely connected with it. These I will now very briefly consider.

The first is by CLERK-MAXWELL* "*On a particular Case of the Descent of a Heavy Body in a Resisting Medium.*" The body is a flat rectangular slip of paper, falling with its longer edges horizontal. It is observed to rotate about an axis parallel to these edges, and to fall in an oblique direction. The motion soon becomes approximately regular; and the deflection of the path from the vertical is to the side towards which the (temporarily) lower edge of the paper slip is being transferred by the rotation. [When the rectangle is not very exact, or the longer edges not quite horizontal, or the slip slightly curved, the appearance, especially when there is bright sunlight, is often like a spiral stair-case.] MAXWELL examines experimentally the distribution of currents, and consequently of pressure, about a non-rotating plane upon which a fluid plays obliquely; and shows that when the paper is rotating the consequent modification of this distribution of pressure tends to maintain the rotation. The reasoning throughout is somewhat difficult to follow, and the circumstances of the slip are very different from those of a ball:—but the direction of the deflection from the unresisted path is always in agreement with the statement made by NEWTON.

Much more intimately connected with our work is a paper by Lord RAYLEIGH† "*On the Irregular Flight of a Tennis Ball,*" in which the "true explanation" of the curved path is attributed to Prof. MAGNUS. The author points out that, in general, the statement that the pressure is least where the speed is greatest, is true only of perfect fluids unacted on by external forces; whereas in the present case the whirlpool motion is directly due to friction. But he suggests the idea of short blades projecting from the ball, the pressure on each of which is shared by the contiguous portion of the spherical surface. Here we have practically NEWTON's explanation—*i.e.* the "pressing and beating of the contiguous air." Lord RAYLEIGH's paper contains an investigation of the form of the stream-lines when a perfect fluid circulates (without molecular rotation) round a cylinder, its motion at an infinite distance having uniform velocity in a direction perpendicular to the axis of the cylinder. And it is shown that the resultant pressure, perpendicular to the general velocity of the stream, has its magnitude proportional alike to that velocity and to the velocity of circulation. [There are some comments on this paper, by Prof. GREENHILL, in the ninth volume of the journal referred to.]

* *Cambridge and Dublin Mathematical Journal*, ix. 145 (1854).

† *Messenger of Mathematics*, vii. 14 (1878).

In the *Beiblätter zu d. Ann. d. Phys.* (1895, p. 289) there appears a somewhat sarcastic notice of my former paper. The Reviewer, evidently annoyed at my remarks on MAGNUS' treatment of ROBINS, which he is unable directly to controvert, refers to HÉLIE, *Traité de Balistique*, as containing an anticipation of my own work. I find nothing there beyond a very small part of what was perfectly well known to NEWTON and ROBINS; except a few of the more immediately obvious mathematical consequences, deduced from the hypothesis (for which no basis is assigned, save that it is the simplest possible) that the transverse deflecting force due to rotation is proportional to the first power of the translational speed.

In the present article I give first a brief account of my recent attempts to determine the initial speed of a golf-ball, and consequently to approximate to the coefficient of v^2 in the assumed expression for the resistance.

Next, instead of facing the labour of the second approximation (suggested in § 10) to the solution of the differential equations, I have attempted by mere numerical calculation to take account of the effect of gravity on the speed of the projectile, and have thus been enabled to give improved, though still rough, sketches of the form of the trajectory when it is not excessively flat. This process furnishes, incidentally, the means of finding the time of passage through any arc of the trajectory.

Third, I treat of the effects of wind, regarded as a uniform horizontal translation of the atmosphere parallel, or perpendicular, to the plane of the path.

Finally, recurring to the limitation of a very flat trajectory, I have treated briefly the effects of gradual diminution of spin during the flight. This loss is shown to be inadequate to the explanation of the unexpectedly small inclination of the calculated path when the projectile reaches the ground. Hence some other mode of accounting for its nearly vertical fall is to be sought, and it is traced to the rapid diminution of the resistance (assigned by ROBINS' law) when the speed has been greatly reduced.

Determination of Initial Speed.

16. The bob of my new ballistic pendulum was a stout metal tube, some 3 feet long, suspended horizontally, near the floor, by two parallel pieces of clock-spring about 2·5 feet apart, and 8·63 feet long. On one end of the tube was fixed transversely a circular disc, 1 foot in diameter, covered with a thick layer of moist clay into which the ball was driven from a distance of 4 feet or so. The whole bob had a mass of about 33 lbs.; and, in the most favourable circumstances, its horizontal displacement was about 3·5 to 4 inches. As the ball's mass is 0·1 lb., the average indicated speed was thus about 200 foot-seconds.* Though I had the assistance of two long drivers, whose

* If l be the length (in feet) of the supporting straps, d the (small) horizontal deflection of the bob, its vertical rise is obviously $d^2/2l$, so that its utmost potential energy is

$$(M+m)gd^2/2l,$$

where M is its mass and m that of the ball. But, if V was the horizontal speed of the ball, that of bob and ball was

habitual carry is 180 yards or upwards, the circumstances of the trials were somewhat unfavourable, for there was great difficulty in hitting the disc of clay centrally. The pendulum was suspended in an open door-way; and heavy matting was disposed all about the clay so as (in ROBINS' quaint language) "to avoid these dangers, to the braving of which in philosophical researches no honour is annexed"; so that the whole surroundings were absolutely unlike those of a golf-course. I therefore make an allowance of 20 per cent., and (as at present advised) regard 240 foot-seconds or something like it as a fair average value of the initial speed of a really well-driven ball:—while thinking it quite possible that, under exceptionally favourable circumstances, this may be increased by 20 or 30 per cent. at least. Now, it is certain that the time of flight is usually about six seconds when the range is about 180 yards:—considerably more for a very high trajectory, and somewhat less for a very flat one. As we have by § 5 the approximate formula

$$t=\frac{a}{V}(\epsilon^{s/a}-1),$$

we may take $a=360$ as a reasonable estimate. This number is possibly some 10 per cent. in error, but it is very convenient for calculation, and golf-balls differ considerably from one another in density as well as in diameter. With it the "terminal velocity" of a golf-ball is about 108 foot-seconds; intermediate to the values deduced from the formulæ of ROBINS and of BASHFORTH, which I make out to be 114 and 95 respectively. With this value of a, it is easy to see that air-resistance, alone, reduces the speed of a golf-ball to half its initial value in a path of 83 yards only. This is the utmost gain of range obtainable (other conditions remaining unchanged) by giving four-fold energy of propulsion! With the value (282) of a deduced from BASHFORTH'S formula, this gain would have been 65 yards only! [So far for the higher speeds, but it is obvious from all ordinary experience of pendulums (with a golf-ball as bob) that slow moving bodies suffer greater resistance than that assigned by this law.]

In passing, I may mention that, on several occasions, I fastened firmly to the ball a long light tape, the further end being fixed (after all twist was removed) to the ground so that the whole was perpendicular to the direction of driving. After the 4 foot flight of the ball, the diameter at first parallel to the tape preserved its initial direction, while the tape was found twisted (in a sense corresponding to underspin) and often through one or two *full* turns, indicating something like 60 or 120 turns per second. This is clearly a satisfactory verification of the present theory.

$mV/(M+m)$. Equating the corresponding kinetic energy to the potential energy into which it is transformed, we find at once $(M+m)gd^2/2l=m^2V^2/2(M+m)$ leading to the very simple expression

$$V=\frac{M+m}{m}d\sqrt{g/l}\,.$$

With the numerical values given in the text we easily find that this is equivalent to

$$V=331\frac{D}{12}1{\cdot}93=53{\cdot}2D\,;$$

where V is, of course, in foot-seconds, but the deflection is now (for convenience) expressed in inches, and called D. Hence the numerical result in the text.

Numerical Approximation to Form of Path.

17. The differential equations of the trajectory were integrated approximately in § 10 by formally omitting the term in g in the first of them, that is so far as the speed is concerned. In other words :—by assuming that ϕ is always very small, or the path nearly horizontal throughout. It was pointed out that if the value of ϕ, thus obtained from the second, were substituted for $\sin\phi$ in the first, equation, we should be able to obtain a second approximation to the intrinsic equation of the path, amply sufficient for all ordinary applications. But the process, though simple enough in all its stages, is long and laborious :—and it is altogether inapplicable to the kinked path, discussed in § 15, which furnishes one of the most singular illustrations of the whole question.

The fact that one of my Laboratory students, Mr JAMES WOOD, had shown himself to be an extremely rapid and accurate calculator led me to attempt an approximate solution of the equations by means of differences :—treating the trajectory as an equilateral polygon of 6-foot sides, and calculating numerically the inclination of each to the horizon, as well as the average speed with which it is described. For we may write the differential equations in the form

$$\frac{1}{2}\frac{d(v^2)}{ds}+\frac{v^2}{a}=-g\sin\phi,$$

$$\frac{d\phi}{ds}=\frac{k}{v}-\frac{g}{v^2}\cos\phi,$$

and these involve approximately

$$v'^2-v^2+2\left(\frac{v^2}{a}+g\sin\phi\right)\delta s=0,$$

$$\phi'-\phi=\left(\frac{k}{v}-\frac{g}{v^2}\cos\phi\right)\delta s.$$

Thus we find, after a six-foot step, the new values

$$v'^2=\left(1-\frac{12}{a}\right)v^2-384\sin\phi,$$

$$\phi'=\phi+\frac{6k}{v}-\frac{192\cos\phi}{v^2}.$$

[If we take account of terms in $(\delta s)^2$, we find that we ought to write for $12/a$ the more accurate expression $12/a\,.\,(1-6/a)$. But this does not alter the *form* of the expression for v'^2. It merely increases by some 2 per cent. the denominator of the coefficient of resistance, of which our estimate is, at best, a very rough one; so that it may be disregarded. But the successive values of v^2 are all on this account too large; and thus the values of ϕ, in their turn, are sometimes increased, sometimes diminished, but only by trifling amounts. This is due to the fact that the change of ϕ depends upon terms having opposite signs; and involving different powers of v, so that their *relative* as well as their *actual* importance is continually changing. These remarks

require some modification when k is such that ϕ may have large values, as for instance in the kinked path treated below. But I do not pretend to treat the question exhaustively, so that I merely allude to this source of imperfection of the investigation.]

Let, now, $a=360$, $k=1/3$, and suppose ϕ to be expressed in degrees. We have, to a sufficient approximation,

$$v'^2=(v^2-400\sin\phi)\left(1-\frac{1}{30}\right),$$

$$\phi'=\phi+\frac{120}{v}-\frac{12000}{v^2}\left(1-\frac{1}{30}\right),$$

and successive substitutions in these equations, starting from any assigned values of v and ϕ, will give us the corresponding values for the next side of the polygon, with the more recent estimate of the coefficient of resistance. See the two last examples in § 19 below, which lead to the trajectories figured as 5 and 6 in the Plate.

Unfortunately, many of Mr WOOD's calculations were finished before I had arrived at my new estimate of the value of a; but their results are all approximately representative of possible trajectories :—the balls being regarded as a little larger, or a little less dense, than an ordinary golf-ball; in proportion as the coefficient of resistance assumed is somewhat too great. And no difficulty arises from the assumption of too great an initial speed; for we may simply *omit* the early sides of the polygon, until we come to a practically producible rate of motion.

18. To discover how far this mode of approximation can be trusted, we have only to compare its consequences with those of the *exact* solution. For the intrinsic equation can easily be obtained in finite terms when there is no rotation. In fact, by elimination of g between the differential equations of § 10, assuming $k=0$, we have at once the complete differential of the equation

$$e^{s/a}v\cos\phi=V\cos\phi_0=V_0 \text{ suppose};$$

where it is to be particularly noticed that V_0 is the speed of the *horizontal component* of the velocity of projection, *not* the total speed. By means of this the second of the equations becomes

$$\frac{d\phi}{ds}=-\frac{g}{V_0^2}e^{2s/a}\cos^3\phi,$$

whence

$$\frac{ag}{V_0^2}(e^{2s/a}-1)=\sec\phi_0\tan\phi_0-\sec\phi\tan\phi+\log\frac{\sec\phi_0+\tan\phi_0}{\sec\phi+\tan\phi}.$$

The following fragments show the nature and arrangement of the results in one of the earlier of Mr WOOD's calculated tables. Having assumed (for reasons stated in the introductory remarks above) that $a=240$, I supplied him with the following formulæ :—

$$v'^2=\left(1-\frac{1}{20}\right)v^2-400\sin\phi(1-0{\cdot}04),$$

$$\phi'=\phi-\frac{12000}{v^2}\cos\phi\,(1-0{\cdot}04),$$

and I took as initial data $V = 300$, $\phi = 15°$; [whence, of course, $V_0^2 = 84{,}000$ nearly. This is required for comparison with the *exact* solution.]

Working from these he obtained a mass of results from which I make a few extracts:—

$s/6$	v^2	v	$1/v$	$\Sigma(1/v)$	ϕ	$\sin\phi$	$\Sigma(\sin\phi)$	$\cos\phi$	$\Sigma(\cos\phi)$
1.	90,000	300	·003	·003	15°	·2588	·2588	·9659	·9659
2.	85,401	292·2	·00342	·00675	14·876	·2565	·5153	·9665	1·9324
3.	81,032	284·6	·00351	·01026	14·746	·2546	·7699	·9671	2·8995
	*		*		*		*		*
20.	33,045	181·8	·00550	·08666	11·028	·1914	4·6102	·9815	19·4569
21.	31,319	177·0	·00565	·09231	10·686	·1854	4·7956	·9826	20·4395
	*		*		*		*		*
40.	11,440	106·9	·00935	·23391	− 1·023	− ·0178	6·6163	·9998	39·3178
41.	10,875	104·3	·00959	·24350	− 2·030	− ·0355	6·5808	·9994	40·3172
	*		*		*		*		*
60.	5453	73·8	·01354	·46935	− 30·748	− ·5113	1·4677	·8595	58·3988
61.	5377	73·3	·01363	·48298	− 32·564	− ·5383	·9294	·8428	59·2416
	*		*		*		*		*

This table gives simultaneous values of s, v, and ϕ directly. t is obviously to be found by multiplying by 6 feet the numbers in column fifth; while by the same process we obtain rectangular coördinates, vertical and horizontal, from the eighth, and the last, columns respectively. Thus for instance we have simultaneously

s	v	t	ϕ	y	x
120	181·8	0^s·52	11°·028	27·66	116·74
240	106·9	1·404	− 1·023	39·69	235·9

(The trajectory is given as fig. 3 in the Plate, and will be further analysed in the next section of the paper.)

From the complete table we find that, in this case, ϕ is positive up to the 38th line inclusive, and then changes sign. It vanishes for $s = 233$ (approximately) after the lapse of 1^s·35. The rectangular coördinates of the vertex are about 230 and 40, and the speed there is reduced to 110. From the exact equation we find $s = 232$ for $\phi = 0°$. This single agreement is conclusive, since the earlier tabular values of s for a given value of ϕ ought to be somewhat in excess of the true values; while the later, and especially those for negative values of ϕ greater than 30° or so, should be somewhat too small:—*i.e.* the calculated trajectory has at first somewhat too little curvature, but towards the end of the range it has too much. It is easy to see that this is a necessary consequence of the mode of approximation employed:—look, for instance, at the fact that the initial speed is taken as constant through the first six feet. See also the remarks in § 17. On the whole, therefore, though the carry may possibly be a little underrated, the numerical method seems to give a very fair approximation to the truth. This admits of easy verification by the help of the value of $d\phi/ds$ last written, for it enables us to calculate the exact value of s for any assigned value of ϕ by a simple difference calculated from the result obtained from an assumed value.

19. Taking the method for what it is worth, the following are a few of the results

obtained from it by Mr Wood. I give the numerical data employed, plotting the curves from a few of the calculated values of x and y. But I insert, at the side of each trajectory, marks indicating the spaces passed over in successive seconds. This would have been a work of great difficulty if we had adopted a direct process, even in cases where the intrinsic equation can be obtained exactly :—and it *must* be carried out when we desire to find the effects of wind upon the path of the ball.

Fig. 1 represents the path when $a=240$ (properly 234), $V=300$, $\phi_0=0°$, and $k=1/3$. This will be at once recognised as having a very close resemblance to the path of a well-driven low ball. The vertex (at 0·76 of the range) and the point of contrary flexure are indicated. This trajectory does not differ very much from that given (for the same initial data) by the roughly approximate formula of § 10; which rises a little higher, and has a range of some ten yards greater. But the assumed initial speed, and consequently the coefficient of resistance, are both considerably too great.

In fig. 2 all the initial data are the same except k, which is now increased to 1/2 :—*i.e.* the spin is 50 per cent. greater than in fig. 1. We see its effect mainly in the increased height of the vertex, and in the introduction of a second point of contrary flexure. A further increase of k will bring these points of contrary flexure nearer to one another, till they finally meet in the vertex, which will then be a cusp, a point of momentary rest, and *the path throughout will be concave upwards*! This is one of the most curious results of the investigation, and I have realized it with an ordinary golf-ball :—using a cleek whose face made an angle of about 45° with the shaft and was furnished with parallel triangular grooves, *biting downwards*, so as to ensure great underspin. [The data for this case give extravagant results when employed in the formula of § 10. The vertex it assigns is 510 feet from the starting point and at nearly 172 feet of elevation: —while the range is increased by 60 or 70 yards. And that formula can never give more than one point of contrary flexure. All this was, however, to be expected; since the formula was based on the express assumption that gravity has no direct effect on the speed of the projectile.]

Fig. 3 shows the result of dispensing altogether with initial rotation, while endeavouring to compensate for its absence by giving an initial elevation of 15°. This figure, also, will be recognised as characteristic of a well-known class of drives; usually produced when too high a tee is employed, and the player stands somewhat behind his ball. Notice, particularly, how much the carry and the time of flight are reduced, though the initial speed is the same. The slight underspin makes an extraordinary difference, producing as it were an unbending of the path throughout its whole length, and thus greatly increasing the portion above the horizon. But of course the pace of the ball, when it reaches the ground, is very much greater than in the preceding cases, it usually falls more obliquely, and it has no back-spin. On all these accounts we should expect to find that the "run" will in general be very much greater. Still, in consequence partly of the greater coefficient of resistance at low speeds, presently to be discussed, over-spin (due to the disgraceful act called "topping") is indispensable for

a really long run. In such a case the carry will, of course, be still further reduced, unless the initial elevation be very considerably increased. (Some of Mr WOOD's numerical results, from which fig. 3 was drawn, were given in the preceding section.)

In fig. 4, α and V are as in fig. 1, but $k=1$ and $\phi_0=45°$. Here we have the kink, of which a provisional sketch (closely resembling the truth) was given in the former instalment of the paper. I have not yet obtained it with a golf-ball, though as already stated I have got the length of producing the cusp above spoken of. But the kink can be obtained in a striking manner when we use as projectile one of the large balloons of thin india-rubber which are now so common. We have only to "slice" the balloon sharply downwards (in a nearly vertical plane) with the flat hand. This is a most instructive experiment, and its repetition presents no difficulty whatever. It is to be specially noticed that, in the particular kink sketched, there is a point of minimum speed somewhat beyond the vertex, and a point of maximum speed, both nearly in the same vertical with the point of projection. The first (where the speed is reduced to 58·7) is reached in a little more than two seconds, the other (where it has risen to 73·8) in rather more than four.

It may be interesting to give a few details of Mr WOOD's calculations for this case:—selecting specially those near the points of maximum and minimum speed, and along with them those for closely corresponding elevations on the ascending side. Also near the vertex. The equations were

$$v_1^2 = v^2\left(1-\frac{1}{20}\right)-400\sin\phi\,(1-0{\cdot}04)$$

$$\phi_1=\phi+\frac{360}{v}-\frac{12000}{v^2}\cos\phi\,(1-0{\cdot}04)$$

$s/6$	v^2	v	$1/v$	$\Sigma(1/v)$	ϕ	$\sin\phi$	$\Sigma(\sin\phi)$	$\cos\phi$	$\Sigma(\cos\phi)$
1.	90000	300	·003	·003	45°	·7071	·7071	·7071	·7071
	*		*		*		*		*
23.	24582	156·8	·00638	·10693	78°·72	·9807	19·6186	·1956	11·3075
	*		*		*		*		*
41.	5583	74·7	·01359	·27640	145°·3	·5693	35·8751	−·8221	6·2814
	*		*		*		*		*
44.	4278	65·4	·01529	·32038	166°·46	·2343	36·9422	−·9722	3·4951
45.	3974	63·0	·01586	·33624	174°·58	·0944	37·0366	−·9955	2·4996
46.	3739	61·1	·01636	·35260	183°·16	−·0553	36·9813	−·9981	1·5015
	*		*		*		*		*
48.	3475	59·0	·01697	·38630	201°·3	−·3633	36·4078	−·9317	− ·5921
49.	3441	58·7	·01704	·40334	210°·5	−·5075	35·9003	−·8616	−1·4537
50.	3464	58·9	·01700	·42034	219°·5	−·6363	35·2640	−·7714	−2·2251
	*		*		*		*		*
67.	5434	73·7	·01357	·67179	313°·1	−·7302	20·0274	·6833	− ·3162
68.	5443	73·8	·01355	·68534	316°·5	−·6880	19·3394	·7258	+ ·4096
69.	5435	73·7	·01357	·69891	319°·9	−·6446	18·6948	·7646	+ ·1742
	*		*		*		*		*

The following data belong to the last elements for which the calculations were made:—

80.	4374	66·1	·01512	·85485	352°·9	−·1224	14·6898	·9925	11·2602
81.	4202	64·8	·01542	·87027	355°·8	−·0732	14·6166	·9973	12·2575

As the last five values of ϕ have been increasing steadily by nearly 3° for each element, it is clear that the direction of motion again rises above the horizontal; but whether the path has next a point of contrary flexure, or another kink, can only be found by carrying the calculation several steps further. [The second kink is very unlikely, as the speed is so much reduced at the point where the calculations were arrested. Mr Wood has gone to Australia, and I had unfortunately told him to stop the numerical work in this particular example as soon as he found that $\Sigma(\cos\phi)$, after becoming negative, had recovered its former maximum (positive) value.]

The trajectories represented in figs. 5 and 6 may be taken as fairly representative of ordinary good play by the two classes of drivers. For we have in both $a=360$, $V=200$. These are the new data, representing (as above explained) the best information I have yet acquired. In fig. 5 $k=1/3$, $\phi_0=10°$; but in fig. 6 $k=0$, $\phi_0=15°$. In spite of its 50 per cent. greater angle of initial elevation, the carry of the non-rotating projectile is little more than half that of the other:—and it takes only one-third of the time spent by the other in the air. But the contrast shows how much more important (so far as carry is concerned) is a moderate amount of underspin than large initial elevation. And we can easily see that initial elevation, which is always undesirable (unless there is a hazard close to the tee) as it exposes the ball too soon to the action of the wind where it is strongest, may be entirely dispensed with. This point is discussed in next section.

On account of their intimate connection with actual practice, I give a few of the numerical results for these two closely allied yet strongly contrasted cases, belonging to two different classes of driving:—choosing sides of each polygon passed at intervals of about 1°, as well as those near the vertices and the point of contrary flexure. The formulæ for these cases are those given at the end of § 17 above:—the second term in the expression for ϕ' being omitted for the latter of the two trajectories.

For Fig. 5.

$s/6$	v^2	v	$1/v$	$\Sigma(1/v)$	ϕ	$\sin\phi$	$\Sigma(\sin\phi)$	$\cos\phi$	$\Sigma(\cos\phi)$
1.	40,000	200	·00500	·00500	10°	·1736	·1736	·9848	·9848
	*		*		*		*		*
25.	15,497	124·5	·00803	·16549	17·552	·3015	6·2345	·9534	25·2200
	*		*		*		*		*
39.	8,216	90·6	·01103	·29869	19·789	·3388	10·7983	·9410	38·4544
	*		*		*		*		*
42.	7,042	83·9	·01192	·33353	19·665	·3366	11·8116	·9417	41·2783
	*		*		*		*		*
54.	3,511	59·3	·01687	·50626	13·611	·2354	15·3925	·9719	52·7246
	*		*		*		*		*
61.	2,387	48·9	·02046	·63904	1·727	·0303	16·3078	·9996	59·6508
62.	2,296	47·9	·02088	·65992	− 0·675	− ·0120	16·2958	·9999	60·6507
	*		*		*		*		*
70.	2,249	47·4	·02109	·83155	− 21·807	− ·3714	14·5533	·9285	68·4117
	*		*		*		*		*
79.	3,157	56·2	·01780	1·00513	− 35·890	− ·5862	9·9647	·8103	76·1309
	*		*		*		*		*
89.	4,338	65·9	·01519	1·16748	− 40·840	− ·6538	3·6521	·7566	83·8830
	*		*		*		*		*
94.	4,853	69·7	·01436	1·24081	− 41·548	− ·6633	0·3507	·7484	87·6381

For Fig. 6.

1.	40,000	200	·00500	·00500	15°	·2588	·2588	·9659	·9659
	*		*		*		*		*
26.	16,035	126·6	·00790	·16507	3·523	·0613	4·5617	·9981	25·5497
	*		*		*		*		*
30.	13,940	118·1	·00847	·19809	0·472	·0082	4·6769	·9999	29·5476
31.	13,472	116·1	·00861	·20670	− 0·360	− ·0064	4·6705	·9999	30·5475
	*		*		*		*		*
44.	9,147	95·6	·01046	·33189	− 13·854	− ·2393	3·0442	·9709	43·4147
	*		*		*		*		*
52.	7,850	88·6	·01129	·41952	− 24·208	− ·4099	·3650	·9121	50·9412

I regret that Mr Wood was obliged to give up his calculations before he had worked out more than about a third of the requisite rows of figures for a trajectory differing initially from fig.5 in the *sole* particular $\phi = 5^\circ$ instead of 10°. This would have been still more illustrative than fig.5 as a contrast with fig.6. But a fairly approximate idea of its form is obtained by taking the earlier part of fig.5, regarded as having the dotted line for its base. See a remark in § 22 below, which *nearly* coincides with this.

Effect of Wind.

20. So far, we have supposed that there is no wind. But with wind the conditions are usually very complex, especially as the speed of the wind is generally much greater at a little elevation than *close* to the ground. Hence I must restrict myself to the case of uniform motion of the air in a horizontal direction. We have in such a case merely to trace, by the processes already illustrated, *the path of the ball relatively to the air*; and thence we easily obtain the path relatively to the earth. Here, of course, it is absolutely necessary to calculate the time of passing through each part of the trajectory relative to the air. If the wind be in the plane of projection, and its speed U, the *relative* speed with which the ball starts has horizontal and vertical components $V\cos\alpha - U$, and $V\sin\alpha$, respectively. Thus, relatively to the moving air, the angle of elevation is given by

$$\tan\alpha' = \frac{V\sin\alpha}{V\cos\alpha - U},$$

and the speed is

$$V' = \sqrt{V^2 - 2UV\cos\alpha + U^2}.$$

The relative trajectory, traced from these data, must now have each of its points displaced forwards by the distance, Ut, through which the air has advanced during the time, t, required to reach that point in the relative path. Of course, for a head-wind, U is negative; and the points of the relative trajectory must be displaced backwards.

Figs. 7, 8, 9 illustrate in a completely satisfactory manner, though with somewhat exaggerated speeds and coefficient of resistance, the results of this process. Mr Wood had calculated for me the path in still air, with $a = 288$ (or, rather, 282), $V = 300$, $\phi = 6^\circ$, $k = 1/3$. Since the time of reaching each point in this path had been incidentally calculated, it had only to be multiplied by 25, and subtracted from the corre-

sponding abscissa, in order to give the actual path when the speed of the head-wind is about 17 miles an hour, and the initial speed about 275. (The exact values of this and of the actual angle of projection must be calculated by means of the preceding formulæ:—but they are of little consequence in so rough an illustration as the present, especially as ϕ_0 and U/V are both small.) The corresponding trajectory is shown in fig.7. If we use the same relative path for wind of 25·5 miles per hour, the actual initial speed must be about 262·5, and the true path is fig.8. Finally, fig.9 gives the result with actual initial speed 250, and head-wind blowing at 34 miles an hour. Here, again, a kink is produced in the actual path, but it is due to a completely different cause from that of fig.4. And it is specially to be noted how much the vertex is displaced towards (and even beyond) the end of the range.

21. It is not necessary to figure the result of a following wind, for such a cause merely lengthens the abscissæ in a steadily increasing ratio, and makes the carry considerably longer, while placing the vertex more nearly midway along the path. But it is well to call attention to a singularly erroneous notion, very prevalent among golfers, viz., that a following wind *carries the ball onwards*! Such an idea is, of course, altogether absurd, except in the extremely improbable case of wind moving faster than the actual initial speed of the ball. The true way of regarding matters of this kind is to remember that there is always resistance while there is relative motion of the ball and the air, and that it is less as that relative motion is smaller; so that it is reduced throughout the path when there is a following wind.

Another erroneous idea, somewhat akin to this, is that a ball rises considerably higher when driven against the wind, and lower if with the wind, than it would if there were no wind. The difference (whether it is in excess or in defect will depend on the circumstances of projection, notably on the spin) is in general very small; the often large apparent rise or fall being due mainly to perspective, as the vertex of the path is brought considerably nearer to, or further from, the player.

These approximations to the effect of wind are, as a rule, very rough; because in the open field the speed of the wind usually increases in a notable manner up to a considerable height above the ground, so that the part of the path which is most affected is that near the vertex. But the general character of the effect can easily be judged from the examples just given.

When the wind blows directly across the path, the same process is to be applied. It is easy to see that the trajectory is no longer a plane curve; and also that, in every case, the carry is increased. But, in general, "allowance is made for the wind," *i.e.* the ball is struck in such a direction as to make an obtuse angle with that of the wind, more obtuse as the wind is stronger. In this case the carry must invariably be shortened. But without calculation we can go little beyond general statements like these.

Effect of Gradual Diminution of Spin.

22. In my former paper I assumed, throughout, that the spin of the ball remains practically unchanged during the whole carry. That this is not far from the truth, is pretty obvious from the latter part of the career of a sliced or a heeled ball. If, however, in accordance with § 4, we assume *it* also to fall off in a geometric ratio with the space traversed:—an assumption which is probable rather than merely plausible; so long, at least, as we neglect the part of the loss which would occur even if the ball had no translatory speed:—the equations of § 10 require but slight modification. For we must now write, instead of k,

$$k\epsilon^{-s/b}.$$

The time rate at which this falls off is proportional to itself and to v, directly, and to b inversely.

If we confine ourselves to the very low trajectories which are now characteristic of much of the best driving, we may neglect (as was provisionally done in § 10) the effect of gravity on the speed of the ball, and write simply

$$v = V\epsilon^{-s/a}.$$

Thus the approximate equation of the path becomes

$$\frac{dy}{dx} = \alpha + \frac{ka'}{V}(\epsilon^{x/a'} - 1) - \frac{ga}{2V^2}(\epsilon^{2x/a} - 1).$$

Here

$$\frac{1}{a'} = \frac{1}{a} - \frac{1}{b}; \text{ and finally}$$

$$y = \alpha x + \frac{ka'^2}{V}(\epsilon^{x/a'} - 1 - x/a') - \frac{ga^2}{4V^2}(\epsilon^{2x/a} - 1 - 2x/a),$$

where α is always very small, perhaps even negative; and may, at least for our present purpose, be neglected. Its main effect is to elevate, or depress, each point of the path by an amount proportional to the distance from the origin; and thus (when positive) it enables us to obtain a given range with less underspin than would otherwise be required.

23. For calculation it is very convenient to begin by forming tables of values of the functions

$$f(p) = \frac{\epsilon^p - 1}{p}, \text{ and } F(p) = \frac{\epsilon^p - 1 - p}{p^2} = \frac{f(p) - 1}{p};$$

for values of p at short intervals from 0 to 3 or so. (Note that the same tables are adaptable to negative values of p, since we have, obviously,

$$f(-p) = \epsilon^{-p} f(p), \text{ and } F(-p) = \epsilon^{-p}(f(p) - Fp)).$$

These we will take for granted. We may now write

$$y = \frac{x^2}{V^2}(kVF(x/a') - gF(2x/a))$$

$$\frac{dy}{dx} = \frac{x}{V^2}(kVf(x/a') - gf(2x/a)),$$

$$\frac{d^2y}{dx^2} = \frac{1}{V^2}(kV\epsilon^{x/a'} - g\epsilon^{2x/a}).$$

The range, and the horizontal distances of the vertex and of the point of contrary flexure, respectively, are given by the values of x which make the second factors vanish:—and it is curious to remark that (to the present rough approximation, of course, and for given values of a and a') these depend only upon the value of kV/g, *i.e.* the initial ratio of the upward to the downward acceleration. Thus so far as the *range* is concerned, the separate values of k and V are of no consequence, all depends on their product. But it is quite otherwise as regards the flatness of the trajectory, for the maximum height is inversely as the square of V. Of course we must remember that one indispensable condition of the approximation with which we are dealing is that the trajectory shall be very flat; and thus, if the range is to be considerable, V cannot be small, and (also of course) k cannot be very large. We have already seen how to obtain a fairly approximate value of a (say 360), but b presents much greater difficulty. We may, therefore, assume for it two moderate, and two extreme values, and compare the characteristics of the resulting paths. If b be infinite, we have the case already treated, in which the spin does not alter during the ball's flight; while, if b be less than a, the spin dies out faster than does the speed and we approximate (at least in the later part of the path) to the case of no spin. Hence we may take for the values of b the following:—∞, 900, 360, and 180:—so that a' has the respective values 360, 600, ∞, and -360. Let the carry ($\bar{x}$) be, once for all, taken as 180 yards. Then, for $y=0$, we must have $2\bar{x}/a = 3$; and the respective values of $\bar{x}/a'$ are 1·5, 0·9, 0, and $-1·5$. With these arguments the values of F are, in order,

1·7873; 0·8807, 0·6908, 0·5, and 0·3258;

so that we have the following approximate values of the ratio kV/g

2·03, 2·59, 3·57, 5·49.

The first two require a moderate amount of spin, only, if we take 240 as the initial speed.

The approximate position of the vertex (x_0) of the first of these paths is given by

$$f(2x_0/a) = 2·03\, f(x_0/a), \text{ or } \epsilon^{x_0/a} = 3·06, \ (x_0/a = 1·1184)$$

whence $x_0 = 402·6$, or about three-fourths of the carry.

The corresponding value of y is about 27 feet.

The point of contrary flexure is at $e^{x/a} = 2\cdot03$, so that $x_1 = 255$, and the value of $\frac{dy}{dx}$ there has its maximum, about 0·07 only.

In the other three paths above, the maximum ordinate and the maximum inclination both increase with the necessarily increased value of k, while the vertex and the point of inflexion both occur earlier in the path. The approximate time of flight, in all, is a little over five seconds. The paths themselves are shown, much foreshortened, in figs. 10, 11, 12, 13, where the unit of the horizontal scale is 3·6 times that of the vertical. This is given with the view of comparing and contrasting them. Fig. 14 shows the first, and flattest, of these paths in its proper form. It is clearly a fair approximation to the actual facts; and when we compare it with the others, as in the foreshortened figures, we see that the assumption of constant spin (§ 4) is probably not far from the truth. For, in the great majority of cases of drives of this character, there is observed to be very little run :—and this can be accounted for only on the assumption that there is considerable underspin left at the pitch. But it is also clear that the falling off of the spin produces comparatively little increase of the obliquity of impact on the ground, even in the exaggerated form in which these paths are drawn. Their actual inclinations to the ground have tangents about 0·49, 0·66, 0·78, and 1·08 respectively. The last, and greatest, of these angles is just over 45°.

24. It is interesting to compare this set of data, and their consequences, with those of §§ 11, 14, 15. The latter were in fair agreement with many of the more easily observed features of a good drive, but they gave too high a trajectory. The new measure of initial speed, and the consequent reduction of the estimated value of the coefficient of resistance, have led to results more closely resembling the truth.

But in all, as we have seen, there is one notable defect. The ball comes down too obliquely, and this is the case more especially when the carry is a long one, and the ball's speed therefore much reduced. I was at first inclined to attribute this to my having assumed the spin to remain constant during the whole flight. This was my main reason for carrying out the investigations described in §§ 22 *sq.* But these give little help, as we have just seen, and I feel now convinced that the defect is due chiefly to the assumption that the resistance is *throughout* proportional to the square of the speed. I intend to construct an apparatus on the principle described in § 16 above, but of a much lighter type, to measure the resistance for speed of 30 feet-seconds or so, downwards. But I shall probably content myself with verifying, if I can, the idea just suggested; leaving to some one who has sufficient time at his disposal the working out of the details when the resistance is proportional (towards the end of the path) to the speed directly, or to a combination of this with the second power. The former is considerably more troublesome than ROBINS' law; and a combination of the two may probably be so laborious as to damp the ardour of any but a genuine enthusiast. The possibility that the law of resistance may change its form for low speeds (*i.e.*, towards and beyond the vertex of the path) throws some doubt upon the accuracy of the determination of the

coefficient of resistance from the range, the time of flight, and the initial speed. But, at present, I have no means of obtaining a more accurate approximation.

25. The whole of this inquiry has been of a somewhat vague character, but its value is probably enhanced, rather than lessened, in consequence. For the circumstances can never be the same in any two drives, even if they are essentially good ones, and made by the same player. To give only an instance or two of reasons for this:—Two balls of equal mass may have considerably different coefficients of resistance in consequence of an apparently trifling difference of diameters, or of the amount or character of the hammering:—or they may have very different amounts of resilience, due to comparatively slight differences of temperature or pressure during their treatment in the mould. The pace which the player can give the club-head at the moment of impact depends to a very considerable extent on the *relative* motion of his two hands (to which is due the "nip") during the immediately preceding two-hundredth of a second, while the amount of beneficial spin is seriously diminished by even a trifling upward concavity of the path of the head during the ten-thousandth of a second occupied by the blow. It is mainly in apparently trivial matters like these, which are placidly spoken of by the mass of golfers under the general title of "knack," that lie the very great differences in drives effected, under precisely similar external conditions, by players equal in strength, agility, and (except to an extremely well-trained and critical eye) even in style.

[*Oct.* 5, 1898.—The printing of this paper has been postponed for nearly three years in the hope, not as yet realised, that I might be able to determine accurately by experiment the terminal speed of an average golf-ball, as well as the average value of k, when (as in § 5) $k\omega v$ represents the transverse acceleration, in terms of the rates of spin and translation. Another object has been to measure the effect of rapid rotation upon the coefficient of resistance to translatory motion. These experiments, in various forms, are still being carried out by means of various modes of propulsion, from a cross-bow to a harpoon-gun. I hope also to procure data, for speed and resistance, applicable to various other projectiles such as cricket-balls, arrows, bird-bolts, etc.]

CPSIA information can be obtained
at www.ICGtesting.com
Printed in the USA
BVHW011810050422
633474BV00002B/25

9 781348 008897

9 781348 008897